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A COMPILATION OF THE MATHEMATICS LEADING TO THE DOUBLET LATTICE METHOD

Max Blair
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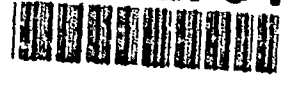

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13. ABSTRACT *(Maximum 200 words)*

This report provides a theoretical development of the doublet lattice method, the method of choice for most subsonic unsteady aerodynamic modelling for over twenty years. This is a tutorial based on many key mathematical developments provided in the References section. An example source code is provided in the Appendix.

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FOREWORD

This report was conducted within the Aeroelasticity Group, Analysis & Optimization Branch, Structures Division (WL/FIBRC), Flight Dynamics Directorate, Wright Laboratory, Wright Patterson AFB Ohio. The work was conducted under Program Element 62201F, Project No 2401, Task 08 and Work Unit 00.

The work was performed during the period of June 1991 through December 1991. Dr Max Blair of the Aeroelasticity Group, WL/FIBRC, was the primary investigator.

The author would like to express his appreciation for the excellent job by Ms Dawn Moore in assembling this report with FrameMaker software.

The author is most appreciative of Dr Dennis Quinn of the Air Force Institute of Technology and Dr Karl G. Guderley for identifying the correct derivation of equation (73). Section VI is roughly based on class notes from the course, "Unsteady Aerodynamics" taught in 1986 by Prof. Marc H. Williams at Purdue University, Indiana. All other sources are identified in the References.

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SECTION I

Introduction

Aircraft flutter is a destructive phenomenon which requires special attention in the design process. The elements of flutter are structural dynamics and unsteady aerodynamics. Of these, it is generally recognized that unsteady aerodynamics are the more difficult to model and the least reliable. In 1935, Theodorsen was the first to develop a practical unsteady incompressible aerodynamic formula[1] for a flutter analysis of a two dimensional airfoil. It was fifty years ago that Smilg and Wasserman of the Aircraft Laboratory of the Wright Air Development Center wrote their landmark report on flutter clearance using the K-method and strip theory. Of course such methods can be addressed with manual calculations. Compressibility is normally associated with the flutter of high speed aircraft. It is impractical to solve the compressible unsteady aerodynamic equations by hand. The doublet lattice method[2] was developed along with improvements in digital computer technology. Hopefully, the doublet lattice method represents the most rudimentary unsteady aerodynamic technique in practice where subsonic compressible flow is a consideration. With the introduction of today's supercomputers, non-linear aerodynamics are now being addressed, in spite of the high cost. It is because of the high cost and technical complications associated with non-linear Computational Fluid Dynamics (CFD) that the doublet lattice method is still used almost exclusively for the subsonic flutter clearance of flight vehicles being designed today. It is difficult to imagine the day when non-linear CFD will replace the doublet lattice method in the preliminary design environment.

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1. Section 5-6 of the work by Bisplinghoff et. al. contains an explanation of Theodorsen's formula.
2. Giesing, Kalman and Rodden

While this document is not a survey report, it is appropriate to acknowledge the original authors of the doublet lattice method, Dr. Edward Albano and Dr. William P. Rodden. The subsequent work of Mr. Joseph P. Giesing, Mrs. Terez P. Kalman, and Dr. William P. Rodden of the Douglas Aircraft Company was sponsored by the Air Force Flight Dynamics Laboratory under the guidance of Mr. Walter J. Mykytow. The two computer codes which resulted from this contractual effort are H7WC[1] and following that, N5KA[2]. These codes are still the de facto standard where the flutter clearance of military aircraft is involved. The geometric options offered in these codes are extensive, including multiple surfaces and slender bodies. The purpose of this document is to derive the fundamental formulae of the doublet lattice method. In order to keep focused on the fundamentals, the formulae derived in this report are restricted to planar wings. The additional work to extend the formulae to wings with dihedral is not conceptually significant. Unsteady aerodynamics over slender bodies is not addressed here.

The mathematical background leading to the doublet lattice method is found among many documents and texts. Considering the importance of the doublet lattice method, it seems surprising that we lack a single consistent derivation. This document attempts to answer the need for a unified derivation of all the important formulae from first principles to the integral formula and also includes a simple doublet lattice source code. The target audience is the graduate student or engineer who has had a first course in aeroelasticity and would like to focus on the mathematics of the doublet lattice method. The author has assumed the reader has a familiarity with the classical topics of potential aerodynamics and linear boundary value problems.

The author takes no credit for developing the formulae. All the work presented here was compiled from many references to create a unified derivation. The author does take credit for any additional illumination which he may cast on

1. Giesing et al, AFFDL-71-5, Vol II, Part I is the original pilot code. It uses doublet panels to model bodies as annular wings.

2. See Giesing et al, AFFDL-71-5, Vol II, Part II is the final deliverable code and uses axial doublets and interference panels for bodies as well as the method of images.

these derivations. The main contribution of this report is that all these derivations are presented in a logical sequence in a single document not available elsewhere. The hope is that the reader will gain an accurate appreciation of the doublet lattice method by following this single derivation.

This report focuses on presenting the general mathematical procedure behind the doublet lattice method. Such mathematics do not make easy reading. The task of making these mathematical derivations pleasurable may be impossible. Learning these mathematics requires that one take pen and paper in hand and derive unfamiliar formulae. The integral formulae of Sections XII and XIII may seem excessively complicated. However, this complication is a matter of bookkeeping and not a matter of high level mathematics beyond undergraduate calculus.

In short, the doublet lattice method is based on the integral equation (276). The integrand of this equation models the effect of the pressure difference (across the plane of the wing) at one wing location on the induced upwash (component of velocity which is normal to the plane of the wing) at another wing location. In a sense, it can be said that equation (276) is entirely equivalent to the linear aerodynamic potential equation (42) and the linearized pressure equation (52). While equation (276) is a specialization of equations (42) and (52), no approximations were assummed in its derivation from the Euler equations.

Euler's five differential equations of inviscid flow are the starting basis for all derivations in this report. These five equations are comprised of one equation of continuity, three equations of momentum and one equation of state. The equations of momentum model pressure equilibrium in each of three coordinate directions. The inviscid restriction of Euler's equations means the momentum equations lack terms of shear force. With no shear force on a fluid, no vorticity (flow rotation) can be developed. While the Euler equations are restricted from generating rotation, they are not restricted from convecting rotation if rotation exists in the initial or boundary conditions to Euler's differential equations. This is the starting assumption in all the subsequent mathematical developments.

The solution to a boundary value problem satisfies both the coupled partial differential equations and the associated boundary condition equations. When a single unique solution exists, the number of variables equals the number of partial differential equations. For the Euler equations, we have five variables and five differential equations. These five equations describe the flow within the domain. The boundary condition is a description of the flow on the boundary of the domain. The important point here is that once Euler's boundary value problem has been stated, all that remains is the mathematical solution. After a general mathematical procedure has been identified, engineers can proceed to make automated applications to their design procedures.

Section II starts with Euler's differential equations with five unknown variables representing flow density, pressure and three components of velocity. Euler's equations are non-linear. The doublet lattice method is linear. The objective of Section II is to reduce Euler's non-linear boundary value problem (five differential equations and associated boundary conditions with five unknown variables) to a linear boundary value problem (one linear differential equation (42) with linear boundary conditions in terms of one unknown variable, the velocity potential). The derivation of this linear (potential) equation follows the approach taken in the text by Bisplinghoff, Ashley and Halfman. The text by Karamcheti provides an excellent explanation of the velocity potential. Equation (42) and the boundary conditions developed in Section IV are sufficient for generating a unique solution for the velocity potential.

Section III assumes the velocity potential of Section II is now a known quantity. The objective of Section III is to develop a single linear differential equation for the unknown pressure variable. After all, the aerodynamicist is interested in the pressure loads on the aircraft, not the velocity potential. The desired linear formula is equation (52). The remainder of Section III provides the derivation of the reverse relation. In other words, given the pressure over the domain, what is the potential over the domain. This relation is equation (73).

If one accepts the linear differential equations (42) and (52) as a linear model for the flow, then Euler's non-linear differential equations are somewhat irrelevant to the subsequent mathematical development of the doublet lattice method. Again, all that remains is the identification of a solution which satisfies the aerodynamic potential equation (42) and the boundary condition.

In Section IV, the non-linear tangential flow boundary condition for Euler's boundary value problem are stated and then linearized The linear boundary condition, together with the linear aerodynamic potential equation (42) form the boundary value problem that will be solved by the doublet lattice method. Finally, the special case of the boundary condition on an oscillating wing is presented. The form of the doublet lattice method presented here assumes the wing, and therefore the flow, oscillate harmonically. Complex notation is assumed here and the reader must be familiar with solving complex algebra problems.

At this point, all the preliminary aspects leading to the doublet lattice method have been completed. The linear boundary value problem has been completely described. The solution procedure begins to take shape in Section V. Mathematicians will typically solve simple linear boundary value problems using the method of separation of variables. This approach is not at all practical for solving the flow over even simple wing planforms. Another method is to identify a set of solutions to the linear aerodynamic potential equation (42). If these solutions can be linearly superimposed such that the boundary conditions are at least approximately satisfied, then the solution is complete.

Simple solutions to the aerodynamic potential equation are not easily identified. This is the motivation for Section V in which we transform the aerodynamic potential equation to the well known acoustic potential equation. We will identify an elementary source solution to this acoustic equation in Section VI. Finally, this solution is modified and transformed back to the coordinates of the aerodynamic potential equation in Sections VI, VII and VIII. Equation (152) is the elementary source solution $\bar{\phi}_s(x, y, z, \xi, \eta, \zeta, t)$ to the aerodynamic

potential equation. The definitions for $\bar{R}$ and τ, found in equation (152) are given in equations (149) and (151). The arguments x, y, z and t in the function $\tilde{\phi}_s$ are identical to the arguments of the potential $\tilde{\phi}(x, y, z, t)$. The arguments ξ, η and ζ represent the x, y and z coordinates of the reference point which labels the elementary source solution. In other words, each source solution $\tilde{\phi}_s(x, y, z, \xi, \eta, \zeta, t)$ represents the potential at coordinates (x, y, z, t) due to a spherically symmetric flow originating from a point located at ξ, η and ζ.

Clones of the elementary source solutions can be placed throughout the flow domain. The resulting potential field is a superposition of the potential arising from each point source as given by equation (152). It turns out that the point sources are placed at coordinates ξ, η, ζ on the boundary of the vehicle. If insufficient point sources are used, the approximating composite potential solution will be "bumpy". In order to provide sufficient smoothness, the approximating solution requires sufficient point sources on the surface. As a matter of fact, one can take the limiting case of a continuum of point sources on the surface. Each differential area of the aerodynamic boundary (excluding the far field boundary) contains a coordinate point identifying another clone of the elementary solution with its own strength. The linearized aerodynamic boundary on a wing is a two dimensional plane or sheet.

Section IX describes a continuous source sheet. A source sheet is a two dimensional surface which has been partitioned into differential areas. Each differential area is assigned a point source of varying flow rate (or strength). In the limit, as the partition is infinitely refined, a continuous source sheet is formed. This purpose of this section is for illustration only. A source sheet will not solve all the boundary conditions for flow over a wing. The potential field associated with a source sheet is the same above and below the plane of the sheet. The pressure field is entirely dependent on the potential field. If the potential is the same above and below, then it is not possible for a single source sheet to generate a pressure difference. However, two opposing source sheets can generate a pressure difference.

Section X describes a continuous doublet sheet. This is the limiting condition as two source sheets are brought together. Each sheet has an opposite strength proportional to the inverse of the distance between them. A single doublet sheet can be used to mathematically model the pressure difference between the upper and lower surfaces of a thin wing. One could use this doublet formulation to solve a boundary value problem for flow over a thin wing. However, this doublet sheet formula is in terms of the (velocity) potential. We are really interested in the pressure load. Therefore, given the solution for the potential, we then have to solve a second problem using differential equation (52) to obtain the pressure field from the potential field. A more direct approach is taken in the remaining sections. Unfortunately, this direct approach increases the complexity of the formulation.

In Section XI, the pressure potential and the acceleration potential are introduced. The pressure potential is the unknown variable of the pressure potential equation (185). The pressure potential equation arises as a direct consequence of the aerodynamic potential equation (42) and the pressure formula (52). The form of the pressure potential equation is identical to the aerodynamic potential equation. Therefore, any elementary solution to the aerodynamic potential equation is also a solution to the pressure potential equation. Therefore, the elementary solution to equation (185) is a source function. Again, the pressure source sheet is symmetric and cannot generate a pressure difference above and below its plane. A pressure doublet solution for oscillatory flow, otherwise known as the acceleration potential Ψ, is developed in equation (191). A pressure doublet sheet can generate a pressure difference. Now that an elementary pressure doublet solution has been identified, it is necessary to formulate the potential field (and then the velocity at the wing boundary) which arises with this pressure. This is given in equation (197).

In Section XII, we use the point pressure doublet solution of Section XI to create a pressure doublet sheet using the same approach taken earlier to expand the point source to a source sheet. This results in integral equation (203) which describes the acceleration potential (pressure) which arises from a pressure

doublet sheet. What we really need and subsequently derive is Equation (224) which is the w component of velocity which arises with a pressure doublet sheet. This is the *integral formula* of Section XII. If one determines a pressure doublet distribution which satisfies the tangential flow boundary conditions using this integral formula, then the boundary value problem has been solved. In a sense, the linear boundary value problem and the specialized integral formula are equivalent. While the concept is simple enough, the procedure for determining the distributed strength of the pressure doublet is not immediately obvious.

Within the integral formula is the kernel function. The kernel function is highly singular (division by zero valued variables of power higher than one) at the surface of the pressure doublet sheet. The purpose of Section XIII is to reduce the severity of the singularity and to put the kernel function in a form which lends itself to numerical evaluation with a computer. (Note: the singularity of the integral formula is not entirely removed.) Section XIII is very detailed and inherently difficult to follow with all the variable substitutions. The kernel function for a planar wing is simply summarized in the following section as equation (277) with supporting equations (278) through equation (281).

The doublet lattice method is a solution procedure for the integral formula. The normal velocity component w is the known boundary condition. What we don't know is the pressure difference Δp across the thin wing. Unfortunately, the unknown pressure falls within the integrand. In the doublet lattice method, the pressure doublet sheet is divided into trapezoidal areas. Within each trapezoid, the unknown pressure function is assumed to take a form with unknown constant coefficients. More specifically, the pressure is assumed spatially constant within the trapezoidal area. The integral formula is evaluated over each trapezoidal area independently. The unknown constant coefficients come out from under the integrand. Some of the remaining singularities in the integrand are avoided by replacing the pressure doublet sheet with a pressure doublet line. Remaining singularities in the resulting line integral (over the doublet line) are addressed using the concept of principle values.

The formulae of Section XIV are encoded in the doublet lattice program of Appendix A as described in Section XV. The reader must keep in mind that the solutions obtained here will not agree exactly with the results of Giesing et. al. because in this report, there have been no steady state corrections. Furthermore, the formulae of Section XIV are restricted to a single wing in the (x, y) plane.

One misconception about the doublet lattice method is that the formulae are made non-singular by replacing the doublet sheet with a lattice structure. This is not the case. The doublet lattice method is made non-singular only by using the concept of principle values. The appeal of the doublet lattice method is solely in its programming simplicity. The assumed form of the pressure function would normally change according to the proximity to the edges of the wing. With the doublet lattice method, the form is the same for all elements. This is a major simplification.

In providing this overview of the doublet lattice method, the reader will hopefully be better prepared to investigate the mathematical details of the following report. Again, the interested reader will find that by using pen and paper while reading this report, he will obtain a special level of ownership of the material presented here. There really is no other way to learn.

SECTION II

From First Principles to the Linearized Aerodynamic Potential Equation

The equations describing an inviscid fluid flow over a solid body will be described in a frame of reference that is attached to the body and travels with it. Here, the three cartesian components are indicated by x, y and z. Time t is a fourth coordinate or dimension. The five state variables are pressure, $p(x, y, z, t)$, density, $\rho(x, y, z, t)$, and the three cartesian components of the velocity with respect to the frame, $u(x, y, z, t)$, $v(x, y, z, t)$, and $w(x, y, z, t)$. A control volume is identified and five appropriate equations are formulated to solve for the five state variables. These are the partial differential equation of continuity, three partial differential equations of momentum and an equation of state. These formulae are presented here without derivation[1].

The continuity equation is given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{1}$$

The three components of the momentum equations are given here. For momentum in the x,y and z directions

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial x} \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial y} \tag{3}$$

1. One is directed to either Chapter 5 of Kuethe and Chow. In the latter, the full Navier-Stokes equations are derived from which the Euler equations are obtained by setting the viscous terms to zero.

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial z} \tag{4}$$

With the following isentropic relation[1], we complete the desired set of five equations which we use to solve for the five unknown state variables u, v w, p and ρ.

$$\frac{p}{\rho^{\gamma}} = \frac{p_o}{\rho_o^{\gamma}} \tag{5}$$

The constant γ is the ratio of specific heats. The variables p_o and ρ_o, are constant reference values of pressure and density. This ratio of pressure and density is constant for any element of fluid. If the entire fluid field originates from a single static reservoir then the ratio of equation (5) is constant for the entire fluid field. The formula for the speed of sound is also presented here without development[2].

$$a^2 = \frac{dp}{d\rho} \tag{6}$$

As a first step in reducing the problem, we denne the fluid velocity vector

$$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k} \tag{7}$$

The condition of irrotationality, $\nabla \times \vec{q} = 0$, allows us to introduce the velocity potential $\Phi(x, y, z, k)$, a new state variable. The relationship between the velocity potential and the three unknown velocity components is presented here without further explanation[3].

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1. Equation (5) is the classical equation of state restricted to isotropic conditions. Such fluid flow is called barotropic. See page 50 of Liepmann and Roshko.

2. See chapter 3.8 of Liepmann and Roshko. Alternatively, see chapter 9.2 of Kuethe and Chow.

3. See page 244 of Karamcheti.

$$\vec{q} = \nabla\Phi = \hat{i}\frac{\partial\Phi}{\partial x} + \hat{j}\frac{\partial\Phi}{\partial y} + \hat{k}\frac{\partial\Phi}{\partial z} \tag{8}$$

With the velocity potential, we can reduce our five equations and five unknowns to three equations and three unknowns. The three unknowns are Φ, p and ρ. These three equations will be (23), (18) and (5).

We denote the magnitude of the velocity as q. It should now be clear from equations (7) and (8) that the relationship between the velocity magnitude and the velocity potential is

$$q = \sqrt{u^2 + v^2 + w^2} = \sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2} \tag{9}$$

We can explicitly state the basis for the irrotational condition $\nabla \times \vec{q} = 0$ in terms of the following three relations. This is obtained using the components of $\vec{q}$ given in equation (7) or equation (8).

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \tag{10}$$

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \tag{11}$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{12}$$

The remainder of this section elaborates on chapter 5.1 of the work by Bisplinghoff Ashley and Halfman (or Dowell et. al.). The three momentum equations (2) - (4) can be put into a single vector equation.

$$\frac{\partial\vec{q}}{\partial t} + [\vec{q}\cdot\nabla]\,\vec{q} = \frac{-\nabla p}{\rho} \tag{13}$$

We now substitute for $\vec{q}$ with the gradient of the velocity potential Φ as defined in equation (8). In addition, equations (10), (11) and (12) are used to obtain equation (14) from equation (13).

$$\frac{\partial}{\partial t}[\nabla\Phi] + \nabla\left[\frac{q^2}{2}\right] = \frac{-\nabla p}{\rho} \tag{14}$$

It is desirable to express the right hand side of equation (14) purely in terms of a gradient operator. As an intermediate step we may see that

$$\frac{\nabla p}{\rho} = \nabla\int_{p_o}^{p}\frac{d\lambda}{\rho(\lambda)} \tag{15}$$

where λ is the dummy or umbral variable of integration. If this is not clear, the explanation follows here. From equation (5) we see that density ρ can be evaluated as a function of pressure alone (independently of x, y, z or t) such that $\rho = \rho(p)$ explicitly. The lower limit of integration, $p_o(t)$ is not spatially variable. The upper limit is $p(x, y, z, t)$. How do we prove that equation (15) is correct? This is shown by using Leibnitz's Rule[1] on the right hand side of equation (15), to obtain the left hand side. Equation (15) is often written in a less rigorous form

$$\frac{\nabla p}{\rho} = \nabla\int\frac{dp}{\rho} \tag{16}$$

Substitute equation (16) into the right hand side of equation (14). Next, reverse the order of integration with respect to time and space on the left hand side of equation (14). Combining all the terms gives

$$\nabla\left[\frac{\partial\Phi}{\partial t} + \frac{q^2}{2} + \int\frac{dp}{\rho}\right] = 0 \tag{17}$$

---

1. Leibnitz's Rule is given here. Keep in mind that λ is an umbral variable and will not be treated as a function of x See page (365) of Hildebrand.

$$\frac{d}{dx}\int_{A(x)}^{B(x)} f(x,\lambda)\,d\lambda = \int_{A(x)}^{B(x)}\frac{\partial f(x,\lambda)}{\partial x}d\lambda + f(x,B)\frac{dB}{dx} - f(x,A)\frac{dA}{dx}$$

We can interpret equation (17) as a single vector equation with three components or as three scalar equations. The differentiated quantity in each of the three scalar equations is identical. Clearly, the three derivatives are zero and therefore, the differentiated quantity is independent of x, y and z. We can obtain a single expression which states this more directly. Integrate each of the three vector components of equation (17) with respect to x, y and z independently. In each case, a constant of integration is added which is independent of x, y and z respectively. Since the same quantity must result from the integration of each of the three vector components, the constant of integration must be independent of x, y and z. The only variable left is t and the constant of integration is a function of time alone. Thus, the integration of (17) leads to the well known Kelvin's (the unsteady version of Bernoulli's) equation.

$$\frac{\partial \Phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho} = F(t) \tag{18}$$

We are left with the task of determining the meaning of the function $F(t)$ which arose as a result of mathematical manipulation and without physical insight. Now, equation (18) must be applicable to the whole flow field and to any part of it. We now specify a far field condition (the region far from the disturbance). Here, the flow is steady and the streamlines are straight. Thus, Φ is time invariant, the pressure is constant and the velocity is assigned a constant magnitude of U. So our far field conditions are $q^2 = U^2$, $dp = 0$ and the derivative $\Phi_t = 0$. By restricting equation (18) to these far field conditions, we obtain

$$F(t) = U^2/2 \tag{19}$$

We can redefine the velocity potential such that

$$\phi = \Phi - \int_0^t F(t)\,dt = \Phi - \frac{U^2 t}{2} \tag{20}$$

This has no effect in the interpretation of the velocity vector and the substitution is made in equation (18). The resulting equation is the modified Bernoulli's equation.

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho} = 0 \tag{21}$$

We momentarily put equation (21) aside and consider the continuity equation. The continuity equation (1) can be put into vector form

$$\frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla \rho) + \rho (\nabla \cdot \vec{q}) = 0 \tag{22}$$

Divide by ρ and substitute $\vec{q} = \nabla \phi$ from equation (8) and equation (20).

$$\frac{1}{\rho}\left[\frac{\partial \rho}{\partial t}\right] + \frac{(\nabla \phi \cdot \nabla \rho)}{\rho} + (\nabla \cdot \nabla \phi) = 0 \tag{23}$$

We now step back and see what we have accomplished. Euler's formulation has been reduced from five equations and five unknowns to a system of three equations and three unknowns. The three equations are, (23), (21) and (5). The three unknown system variables are ϕ, p and ρ. While the three velocity components no longer appear, they can be obtained from the potential ϕ. We still have the goal of formulating a single equation in terms of a single system variable and independent of the others. We will choose the velocity potential ϕ as this single system variable. The other variables, p and ρ, can be obtained subsequently to the solution for ϕ.

In attaining this single equation, we start with equation (23) which has three parts. The third part is already a function of ϕ. The first part can be made a function of ϕ with the following manipulations based on equations (5) and (6) and Leibnitz's Rule.

$$\left[\frac{a^2}{\rho}\right]\frac{\partial \rho}{\partial t} = \left[\frac{1}{\rho}\right]\frac{dp}{d\rho}\frac{\partial \rho}{\partial t} = \left[\frac{1}{\rho}\right]\frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\int_{p_o}^{p} \frac{d\lambda}{\rho(\lambda)} = \frac{\partial}{\partial t}\int \frac{dp}{\rho}$$

Interchanging the first and last steps gives a useful intermediate formula.

$$\frac{\partial}{\partial t}\int\frac{dp}{\rho} = \left[\frac{a^2}{\rho}\right]\frac{\partial \rho}{\partial t} \tag{24}$$

The left hand side of equation (24) employs abbreviated notation which is consistent with the meaning used earlier in equation (16). We see where the right hand side of equation (24) fits in a time differentiated form of equation (21). Making the substitution in equation (21) we obtain

$$\frac{a^2}{\rho}\left[\frac{\partial \rho}{\partial t}\right] = -\frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial t} + \frac{q^2}{2}\right]$$

Divide by a^2 to obtain

$$\frac{1}{\rho}\left[\frac{\partial \rho}{\partial t}\right] = \left[\frac{-1}{a^2}\right]\frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial t} + \frac{q^2}{2}\right] \tag{25}$$

Equation (25) can be substituted into the left hand side of equation (23). Finally, the second term of equation (23) can also be put in terms of ϕ with the following manipulations. Starting with equation (21) we obtain

$$-\nabla\left[\frac{\partial \phi}{\partial t} + \frac{q^2}{2}\right] = \nabla\int\frac{dp}{\rho} \tag{26}$$

First, using equation (16) and then equations (5) and (6), we see that equation (26) can also be written as

$$-\nabla\left[\frac{\partial \phi}{\partial t} + \frac{q^2}{2}\right] = \frac{\nabla p}{\rho} = \frac{1}{\rho}\frac{dp}{d\rho}\nabla\rho = \frac{a^2}{\rho}\nabla\rho$$

Interchanging the first and last step and dividing by a^2 gives

$$\frac{\nabla \rho}{\rho} = \left[\frac{-1}{a^2}\right]\nabla\left[\frac{\partial \phi}{\partial t} + \frac{q^2}{2}\right] \tag{27}$$

Finally, taking the dot product with $\nabla\phi$, it follows that

$$\frac{(\nabla\phi \cdot \nabla\rho)}{\rho} = \left[\frac{-1}{a^2}\right]\nabla\phi \cdot \nabla\left[\frac{\partial\phi}{\partial t} + \frac{q^2}{2}\right] \tag{28}$$

This is in the desired form, ready to substitute into equation (23). But, first we carry out the operations on the right hand side of equation (28) to give

$$\frac{(\nabla\phi \cdot \nabla\rho)}{\rho} = \frac{-1}{a^2}\left[\nabla\phi \cdot \frac{\partial}{\partial t}(\nabla\phi) + \nabla\phi \cdot \nabla\left(\frac{q^2}{2}\right)\right]$$

Now, use equations (8) and (9) and note that $(q^2 = \nabla\phi \cdot \nabla\phi)$

$$\frac{(\nabla\phi \cdot \nabla\rho)}{\rho} = \frac{-1}{a^2}\left[\frac{\partial}{\partial t}\left[\frac{q^2}{2}\right] + \nabla\phi \cdot \nabla\left[\frac{q^2}{2}\right]\right] \tag{29}$$

Equations (25) and (29) can be substituted into equation (23) to give the full potential equation.

$$\nabla^2\phi - \frac{1}{a^2}\left[\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(q^2) + \nabla\phi \cdot \nabla\left[\frac{q^2}{2}\right]\right] = 0 \tag{30}$$

Equation (30) is almost in the desired form of a single equation in ϕ alone. We address the parameter, $a = a(x, y, z, t)$ in the following development. Starting with equation (21), we substitute for $p(\rho)$.

$$\frac{\partial\phi}{\partial t} + \frac{q^2}{2} = -\int_{p_o}^{p}\frac{dp}{\rho} = -\gamma\int_{\rho_o}^{\rho}\rho^{(\gamma-2)}d\rho = \left[\frac{-\gamma}{\gamma-1}\right][\rho^{(\gamma-1)}]_{\rho_o}^{\rho} \tag{31}$$

Using equations (5) and (6), we see $a^2 = \gamma\rho^{(\gamma-1)}$ and substitution in equation (31) gives the following result.

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} = -\left(\frac{a^2 - a_o^2}{\gamma - 1} \right) \tag{32}$$

Here, $a_o(t)$ is the speed of sound associated with the reference (e.g.: the far field) conditions $p = p_o$ and $\rho = \rho_o$. So, by equation (32) we have a formula for a in terms of ϕ. This formula for a can be used in equation (30) to obtain the one single equation in ϕ. However, the form of this equation is complicated and we don't solve it anyway. At this point, we assume a is somehow restricted and address this issue later. We expand equation (30) using subscripts to denote differentiation.

$$\nabla^2 \phi - \left[\frac{1}{a^2} \right] \phi_{tt} - \left[\frac{2}{a^2} \right] (\phi_x \phi_{xt} + \phi_y \phi_{yt} + \phi_z \phi_{zt})$$

$$- \left[\frac{1}{a^2} \right] (\phi_x^2 \phi_{xx} + \phi_y^2 \phi_{yy} + \phi_z^2 \phi_{zz})$$

$$- \left[\frac{1}{a^2} \right] (2\phi_x \phi_y \phi_{xy} + 2\phi_x \phi_z \phi_{xz} + 2\phi_y \phi_z \phi_{yz}) = 0 \tag{33}$$

The steady state form of equation (33) is obtained by setting to zero the derivatives with respect to t. We proceed under the assumption that a steady state solution to this non-linear equation exists.

We now turn our attention back to equation (33). The unknown ϕ, is divided into two components, a steady state component (bar) and a small disturbance component (tilde) which is time dependent. Likewise, p and ρ will be divided into a steady state component and a small disturbance component (tilde).

$$\phi(x, y, z, t) = \bar{\phi}(x, y, z) + \tilde{\phi}(x, y, z, t) \tag{34}$$

$$p(x, y, z, t) = \bar{p}(x, y, z) + \tilde{p}(x, y, z, t) \tag{35}$$

$$\rho(x, y, z, t) = \bar{\rho}(x, y, z) + \tilde{\rho}(x, y, z, t) \tag{36}$$

The speed of sound is assumed time invariant in this linearization process. So we denote this restriction as

$$a(x, y, z, t) = \bar{a}(x, y, z) \tag{37}$$

(Ultimately, we will keep a constant. One will see that a higher order approximation has no effect on the resulting linear formulation.) We substitute equations (34) and (37) into equation (33) and delete any non-linear terms in ϕ and its derivatives. Furthermore, we subtract out the steady state condition. We obtain the following time linear partial differential equation (PDE).

$$(\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz}) - \left[\frac{1}{\bar{a}^2}\right][\tilde{\phi}_{tt} + 2\bar{\phi}_x\tilde{\phi}_{xt} + 2\bar{\phi}_y\tilde{\phi}_{yt} + 2\bar{\phi}_z\tilde{\phi}_{zt} +$$

$$\bar{\phi}_x^2\tilde{\phi}_{xx} + \bar{\phi}_y^2\tilde{\phi}_{yy} + \bar{\phi}_z^2\tilde{\phi}_{zz} + 2(\bar{\phi}_x\bar{\phi}_y\tilde{\phi}_{xy} + \bar{\phi}_y\bar{\phi}_{xy}\tilde{\phi}_x + \bar{\phi}_x\bar{\phi}_{xy}\tilde{\phi}_y) +$$

$$2(\bar{\phi}_x\bar{\phi}_z\tilde{\phi}_{xz} + \bar{\phi}_z\bar{\phi}_{xz}\tilde{\phi}_x + \bar{\phi}_x\bar{\phi}_{xz}\tilde{\phi}_z) +$$

$$2(\bar{\phi}_y\bar{\phi}_z\tilde{\phi}_{yz} + \bar{\phi}_z\bar{\phi}_{yz}\tilde{\phi}_y + \bar{\phi}_y\bar{\phi}_{yz}\tilde{\phi}_z)\ ] = 0 \tag{38}$$

Here, we can clarify our restriction on the speed of sound a. If we had assumed that a was of a higher order, the high order terms would have dropped out when we dropped all the non-linear terms.

While partial differential equation (38) is linear in $\tilde{\phi}$, the solution for $\tilde{\phi}$ is still difficult to analyze for any general description of the steady state field $\bar{\phi}(x, y, z)$. This spatially variable $\bar{\phi}(x, y, z)$ results in spatially variable coefficients in this PDE for $\tilde{\phi}$. Certainly, there are no elementary solutions available for the entire flow field described in equation (38). So, we choose to further restrict our PDE to simple steady mean flows. If we let the steady mean flow have a uniform velocity of U with streamlines in the x direction, then for the entire flow field, the coefficients are simply

$$\bar{\phi}(x, y, z) = Ux \tag{39}$$

$$\bar{a}(x, y, z) = a_o \tag{40}$$

where a_o is the constant value of a in the far field.

Substituting steady-state components given by equations (39) and (40) into equation (38), we obtain

$$(\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz}) - \left(\frac{1}{a_o^2}\right)(\tilde{\phi}_{tt} + 2U\tilde{\phi}_{xt} + U^2\tilde{\phi}_{xx}) \tag{41}$$

By collecting similar terms we produce the classical linear small disturbance velocity potential PDE. We now name this equation the aerodynamic potential equation given here as equation (42). We have defined the Mach number as $M = U/a$. Both the steady velocity U and steady speed of sound a were assumed to be constant throughout the flow field in the process of arriving at linear equation (41). For a non-linear solution, these quantities would not be constant. It follows that for linear small disturbance theory, the Mach number is assumed constant for the entire flow field. Later, we will use the notation, $\beta^2 = 1 - M^2$.

$$(1 - M^2)\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz} - \left(\frac{2U}{a_o^2}\right)\tilde{\phi}_{xt} - \left(\frac{1}{a_o^2}\right)\tilde{\phi}_{tt} = 0 \tag{42}$$

Now, all the coefficients are constant and we can identify elementary solutions. This will be assumed to be the governing PDE for describing the aerodynamic behavior. The boundary value problem is comprised of equation (42) and the linear boundary conditions to be specified in Section IV. It can be shown that the solution to this boundary value problem is unique.

SECTION III

The Linearized Pressure Equation

We can solve for $\tilde{\phi}(x, y, z, t)$ using the boundary value problem comprised of the aerodynamic potential equation (42) and the boundary conditions to be described later. However, we are really interested in the pressure. For this reason, we use Bernoulli's equation (21) to develop a linear expression for p as a function of $\tilde{\phi}$. In other words, having developed a boundary value problem for $\phi(x, y, z, t)$, independent of the pressure p, we now develop a linear formula for determining pressure. This functional relationship will be given as equation (52). Furthermore, we will express $\tilde{\phi}$ as a function of p in equation (73).

Using equation (5) we can write pressure as an explicit function of density.

$$p = \left(\frac{p_o}{\rho_o^\gamma}\right)\rho^\gamma \tag{43}$$

Thus the integral expression in equation (21) can be written

$$\int_{p_o}^{p}\frac{dp}{\rho} = \frac{\gamma p_o}{\rho_o^\gamma}\int_{\rho_o}^{\rho}\rho^{\gamma-2}d\rho \tag{44}$$

Carrying out the integration on the right hand side of equation (44), we obtain

$$\int_{p_o}^{p}\frac{dp}{\rho} = \left[\frac{-\gamma}{\gamma-1}\right]\left[\frac{p_o}{\rho_o}\right] + \left[\frac{\gamma}{\gamma-1}\right]\left[\frac{p^{1/\gamma}}{\rho_o}\right]p^{\left[\frac{\gamma-1}{\gamma}\right]} \tag{45}$$

Equation (45) can be substituted into integral equation (21) to obtain the non-linear algebraic expression for pressure in terms of ϕ alone. In equations (35) and (36) we introduced the small disturbance notation for pressure and density. Here, we will linearize about the constant far field pressure condition $\bar{p}(x, y, z) = p_o$ and density $\bar{\rho}(x, y, z) = \rho_o$. Since we are interested in linear aerodynamics, we use the linear part of a Taylor equation identified here for some function $F(p)$ synonymous with the right hand side of equation (45).

$$\int_{p_o}^{p} \frac{dp}{\rho} = F(p) = F(p_o) + F'(p_o)(p - p_o) + hot \tag{46}$$

Carrying out these operations on equation (45) and simplifying gives

$$\int_{p_o}^{p} \frac{dp}{\rho} = \left[\frac{1}{\rho_o}\right](p - p_o) = \left[\frac{1}{\rho_0}\right]\tilde{p} \tag{47}$$

This linear expression can be substituted into equation (21). The derivative ϕ_t in equation (21) is easily linearized as

$$\frac{\partial \phi}{\partial t} \approx \frac{\partial \tilde{\phi}}{\partial t} \tag{48}$$

Finally, the term $q^2/2$ in equation (21) is linearized about the free stream velocity components $u = U$, $v = 0$, and $w = 0$. Starting with

$$\frac{q^2}{2} = \frac{1}{2}(u^2 + v^2 + w^2) \tag{49}$$

We use a multi-variable Taylor series expansion on equation (49) to obtain

$$\frac{q^2}{2} \approx \frac{1}{2}U^2 + U(u - U) = \frac{1}{2}U^2 + U\tilde{\phi}_x \tag{50}$$

We substitute equation (47), (48) and (50) into Bernoulli's equation (21) to obtain the linearized expression

$$(p - p_o) = -\rho_o\left[\tilde{\phi}_t + U\tilde{\phi}_x + \frac{1}{2}U^2\right] \tag{51}$$

It may appear that this expression is not satisfied at the far field condition where $\tilde{\phi}_t = 0$, $\tilde{\phi}_x = 0$ and $p = p_o$. This is easily explained. In the development of equation (21), we defined ϕ as an alternative definition of the velocity potential Φ. In doing so, we essentially set the far field pressure to zero. If p_o is to describe the actual far field pressure, then we subtract the constant $\rho_o (U^2/2)$ factor and we obtain the result

$$(p - p_o) = -\rho_o (\tilde{\phi}_t + U\tilde{\phi}_x) \tag{52}$$

It turns out the $\rho_o U^2/2$ factor will be inconsequential because we will be primarily interested in the pressure differences between the upper and lower surface of a wing.

Equation (52) is a very important formula. In Section II, a linear boundary value formulation for the potential $\tilde{\phi}(x, y, z, t)$ was derived from Euler's equations. Once the potential function $\tilde{\phi}(x, y, z, t)$ is determined, equation (52) is used to determine the pressure $p(x, y, z, t)$. Later in this report, it will be useful to have a formula for the potential given the pressure. The following development[1] achieves this formula as equation (73).

We start with equation (52). We temporarily[2] define $\bar{p} = (p_o - p)/\rho_o$.

$$\bar{p} = \tilde{\phi}_t + U\tilde{\phi}_x \tag{53}$$

We use the method of characteristics to solve this problem. This method utilizes a coordinate transformation such that equation (53) becomes an ordinary differential equation which can be easily integrated. The new coordinates are

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1. See the reference by Colton.

2. $\bar{p}$ was also used earlier to define the steady state pressure. This previous definition is not to be confused with this new temporary use of the symbol $\bar{p}$.

$$\xi = \xi(x, t) \tag{54}$$

$$\eta = \eta(x, t) \tag{55}$$

The potential function and the pressure function are given new designations in the new coordinate frame. (Do not confuse ψ here with the acceleration potential of Section XI. No connection is intended.)

$$\tilde{\phi}(x(\xi, \eta), t(\xi, \eta)) = \psi(\xi, \eta) \tag{56}$$

$$\bar{p}(x, t) = \hat{p}(\xi, \eta) \tag{57}$$

We ignore the role of the y and z coordinates. It will be seen that they have no influence on the final solution. By the chain rule, we have from equation (56).

$$\tilde{\phi}_t = \psi_\xi \xi_t + \psi_\eta \eta_t \tag{58}$$

$$\tilde{\phi}_x = \psi_\xi \xi_x + \psi_\eta \eta_x \tag{59}$$

Substituting equations (57), (58) and (59) into equation (53), we obtain

$$\hat{p} = (\psi_\xi \xi_t + \psi_\eta \eta_t) + U(\psi_\xi \xi_x + \psi_\eta \eta_x) \tag{60}$$

Rearranging the terms gives

$$\hat{p} = \psi_\xi (\xi_t + U\xi_x) + \psi_\eta (\eta_t + U\eta_x) \tag{61}$$

We are free to choose the relationship between the (ξ, η) coordinates and the (x, y) coordinates. So we now specify

$$(\eta_t + U\eta_x) = 0 \tag{62}$$

This is satisfied if we simply choose

$$\eta(x, t) = x - Ut \tag{63}$$

Equation (61) becomes

$$\hat{p} = \psi_\xi (\xi_t + U\xi_x) \tag{64}$$

If we specify ξ such that

$$\xi = x \tag{65}$$

we obtain the relationship between pressure and the potential.

$$\hat{p} = U\psi_{\xi} \tag{66}$$

Thus, equation (61) has been reduced to equation (66). Now we simply integrate equation (66). The constant of integration is chosen at $\xi = x = -\infty$, the far upstream condition.

$$\psi(\xi, \eta) = \frac{1}{U}\int_{-\infty}^{\xi} \hat{p}(\lambda, \eta)\, d\lambda \tag{67}$$

Now we work to cast this expression in the original coordinates. From equations (63) and (65) we have the inverse relations

$$x = \xi \tag{68}$$

$$t = \frac{\xi - \eta}{U} \tag{69}$$

We make the change from $\hat{p}$ to $\bar{p}$ in equation (67).

$$\psi(\xi, \eta) = \frac{1}{U}\int_{-\infty}^{\xi} \bar{p}\left[\lambda, \frac{\lambda - \eta}{U}\right] d\lambda \tag{70}$$

Now

$$\tilde{\phi}(x, t) = \psi(\xi, \eta) = \psi(x, x - Ut) \tag{71}$$

Therefore, by substituting equation (71) into equation (70), we obtain

$$\tilde{\phi}(x, t) = \frac{1}{U}\int_{-\infty}^{x} \bar{p}\left[\lambda, \frac{\lambda - x + Ut}{U}\right] d\lambda \tag{72}$$

By replacing $\bar{p} = (p_0 - p)/\rho_o$ and rearranging terms, we obtain the desired result

$$\tilde{\phi}(x,t) = \frac{1}{\rho_o U}\int_{-\infty}^{x}\left[p\left[\lambda, t-\frac{x-\lambda}{U}\right]-p_o\right]d\lambda \tag{73}$$

In this formula, the reader is reminded that λ is the dummy variable of integration representing integration in the x direction. We dropped the y and z dependence as a matter of convenience. Adding y and z back to the argument list of the pressure function $p(x,t) \Rightarrow p(x,y,z,t)$ in the integrand will not change the integral formula. In other words, equation (73) for $\tilde{\phi}(x,y,z,t)$ is written as

$$\tilde{\phi}(x,y,z,t) = \frac{1}{\rho_o U}\int_{-\infty}^{x}\left[p\left[\lambda, y, z, t-\frac{x-\lambda}{U}\right]-p_o\right]d\lambda$$

Again, this formula is used if we know $p(x,y,z,t)$ and we need the corresponding function $\phi(x,y,z,t)$. Now λ appears twice in the argument list, once in the x and once in the t place holders of the pressure function $p(x,y,z,t)$. This has the influence that as we integrate downstream from $x = -\infty$, we evaluate the pressure at time prior to t. Later in this report, we will refer to this earlier time in the integrand as retarded time. Therefore, if we know the pressure for all time prior to time t, we can evaluate the above integral expression for the potential $\tilde{\phi}(x,y,z,t)$.

SECTION IV

Linearized Boundary Conditions from First Principles

The aerodynamic potential equation (42) was formulated in section (II). It can be shown that the solution to this partial differential equation is unique given the appropriate boundary conditions. The boundary condition specifies the potential or a directional derivative of the potential on all surfaces which define the computational domain. The directional derivative of the velocity potential is a component of the velocity vector. So we specify either the potential or a component of the velocity on the surface of the computational domain.

When the flow over a flight vehicle is addressed, the computational domain is defined interiorly by the surface of the flight vehicle and the trailing wake and exteriorly by the far field conditions. The domain may be nominally fixed with respect to the vehicle body of interest. Certainly, this is usually the case for most aerodynamic developments and this has been the case in this text. However, one may have a reason to attach the frame of reference to the atmosphere and let the vehicle pass through the reference frame. For this special case, the interior surface of the computational domain moves and therefore, the boundary condition is applied over a moving surface. Here, we will only address boundary conditions on a surface which is nominally fixed with respect to the frame of reference.

Vehicle Surface Boundary Condition

A time variant surface in three space can be described by the equation

$$F(x, y, z, t) = 0 \tag{74}$$

Of course, it is unrealistic to formulate a closed form expression for the function F if the surface is detailed as may be the case for a flight vehicle. However, one can always formulate the function F for a sufficiently small patch of the total surface.

The boundary condition on the vehicle specifies that the flow is tangential to the surface. In other words, there is no component of flow normal to the surface. Mathematically, this is described by[1]

$$\frac{\partial F}{\partial t} + \vec{V} \cdot \nabla F = 0 \tag{75}$$

This equation can be linearized about any reference shape. We have linearized the potential equation (33) about an undisturbed uniform flow as described by equation (39). Thus, the boundary condition will be linearized in kind, about an undisturbed and uniform flow. As mentioned earlier, this is a severe restriction. Basically, this limits us to modelling flow disturbances over slender bodies and thin wings.

Here, we will linearize equation (75) specifically for a thin wing. We denote the functional description of the surface of the wing as $F = F_w(x, y, z, t)$. This function is now constrained to two uncoupled components, the deformation of the midplane h_m and the thickness envelope h_t about the undeformed midplane. The undeformed midplane is conveniently designated as the $z = 0$ plane. This is stated mathematically as

$$F_w(x, y, z, t) = z - h_m(x, y, t) \pm h_t(x, y, t) = 0 \tag{76}$$

For the linearized flow about a uniform free stream, $\vec{V} = U\hat{i}$, the aerodynamic velocity vector is mathematically described as

$$\vec{V} = (U + u)\,\hat{i} + (v)\,\hat{j} + (w)\,\hat{k} \tag{77}$$

1. Karamcheti, pp 191

For the remainder of this text, we redefine u, v, and w differently than in equation (7) to represent the small disturbance from the uniform free stream. We substitute equations (76) and (77) into equation (75) and denote $h = h_m \pm h_t$.

$$-\frac{\partial h}{\partial t} - (U+u)\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} + w = 0 \tag{78}$$

We desire a linear relation between the velocity components at the surface of the wing and the function $h(x, y, t)$. So, we make equation (78) linear in h, u, v and w (keeping in mind that the derivative is a linear operator) by dropping the non-linear terms. Equation (78) now becomes

$$w = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \tag{79}$$

It is now clear that the components of $(h = h_m \pm h_t)$ can be treated independently in the linearized boundary conditions. This is especially important when the dynamic response of a wing is considered. Here, we normally assume the thickness effects are not time dependent.

$$h(x, y, t) = h_m(x, y, t) \pm h_t(x, y) \tag{80}$$

Therefore, when analyzing the dynamic response of a wing, we superimpose the dynamic response due to h_m on a separate time invariant solution using h_t. This has an important influence in our choice of the doublet sheet to model the aerodynamics for the time dependent flow over the wing midplane and our choice of source panels to model the aerodynamics for the time invariant component of flow the wing thickness envelope. The concepts of a doublet sheet and a doublet lattice will be discussed later.

The Far Field Boundary Condition

The far field boundary condition is enforced at far distances from the interior boundary where the flow is uniform. It will be clear that the far field condition is satisfied automatically when one uses a superposition of sources or doublets on the interior boundary. The influence of sources and doublets dies out at infinite distances.

The Trailing Wake Boundary Condition

Steady state lift cannot be sustained where there is no viscosity. However, the aerodynamic potential equation (42) is restricted to irrotational and therefore inviscid flow. This contradiction is corrected by fixing the flow circulation about the airfoil to meet the Kutta condition. The Kutta condition specifies a smooth and finite flow off the sharp tailing edge of a lifting surface in incompressible flow. The velocity vector is not allowed to deflect as the flow passes over the trailing edge. If it does deflect, the velocity becomes locally infinite. This trailing edge condition and wake are completely characterized for incompressible flow[1].

Linearized steady compressible flow over planar wings can be transformed to the incompresible case (using the Prandtl-Glauert transformation) and therefore the trailing edge condition is well understood. For linearized unsteady compressible flow, the trailing edge condition is not as clearly characterized. For instance, at high frequency, it is experimentally known that the flow off the trailing edge is not tangential. This is an important topic and warrents further study[2]. *For linearized flow, we introduce a trailing wake or sheet with the property that the pressure difference across the sheet is zero.* This is the only condition imposed on the wake. We allow a non-tangential flow on the wake. A few of the consequences of these assumptions are now discussed.

For our linearized boundary value problem, the wing and therefore the trailing edge are in the $z = 0$ plane. In time, the mathematical representation of the wing slices out a plane region as it moves forward through the air at velocity U. We will treat this planar region as the trailing wake. (We could have a non-planar wake. However, for our linearized system of equations, the added accuracy is not warrented.) It is assummed that there is no jump in the non-zero velocity component, $w = \phi_z$ across the wake. Therefore, ϕ_z is symmetric with respect to z in its first approximation (eg. Fourier Series) with respect to z . With ϕ_z symmetric, it follows that ϕ, ϕ_x and ϕ_t are all antisymmetric with respect to z in the first approximation. Therefore the pressure

1. See Sections 13-8 through 13-10 of Karamcheti for a discussion of the wake. Chapter 7 discusses flow discontinuity.
2. The work by Guderley partially addresses this topic.

$$p = \phi_t + U\phi_x \tag{81}$$

is antisymmetric with respect to z in the vicinity of the wake. Since pressure jumps are not admissible, it follows from antisymmetry that the pressure is zero in the plane of the wake.

The trailing wake boundary condition for steady flows requires ϕ to be antisymmetric across the wake. We allow the possibility of an anti-symmetric jump in ϕ across the wake. While continuity in pressure is a requirement, continuity in ϕ is not. We are free to use the wake as a boundary on the domain of the potential field. Since ϕ is antisymmetric and discontinuous across the wake boundary, so is ϕ_x.

Boundary Conditions on an Oscillating and Deforming Wing

For simple wings, it is often expedient to represent the deformations in terms of polynomials in space and harmonic in time. Certainly, this is the case for flow over a plate. We assume a polynomial of order n_x in x and n_y in y. Frequency is denote as ω and we use complex notation. So we constrain the out-of-plane deformation to the following series expression:

$$h(x, y, t) = \left[\sum_{j=0}^{n_x}\sum_{k=0}^{n_y} a_{jk}x^j y^k\right]e^{i\omega t} \tag{82}$$

Of course, we really mean to equate h with the real part of the right hand side of equation (82). We substitute equation (82) into equation (79) to obtain

$$w = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}$$

$$= \left(i\omega\left[\sum_{j=0}^{n_x}\sum_{k=0}^{n_y} a_{jk}x^j y^k\right] + U\left[\sum_{j=0}^{n_x}\sum_{k=0}^{n_y} j a_{jk}x^{j-1} y^k\right]\right)e^{i\omega t} \tag{83}$$

The complex modulus of w is denoted as $\bar{w}$ such that $w = \bar{w}e^{i\omega t}$. From equation (83) we obtain

$$\bar{w} = i\omega \sum_{k=0}^{n_y} a_{ok} y^k + \left[\sum_{j=1}^{n_x} \sum_{k=0}^{n_y} a_{jk} (i\omega x^j y^k + U j x^{j-1} y^k) \right] \tag{84}$$

The main reason for developing equation (84) is to provide an example of the boundary condition formulation which may be used in the doublet lattice method. Input for the example doublet lattice program in Appendix A is in this form.

SECTION V

Transformation to the Acoustic Potential Equation

In Section II, we started with the Euler equations and derived the aerodynamic potential equation.

$$(1-M^2)\,\tilde{\phi}_{xx}+\tilde{\phi}_{yy}+\tilde{\phi}_{zz}-\left[\frac{2U}{a^2}\right]\tilde{\phi}_{xt}-\left[\frac{1}{a^2}\right]\tilde{\phi}_{tt}=0 \tag{85}$$

We now show the relationship between equation (85) and the classical acoustic potential equation.

$$\underset{\sim}{\phi}_{x_o x_o}+\underset{\sim}{\phi}_{y_o y_o}+\underset{\sim}{\phi}_{z_o z_o}-\left[\frac{1}{a^2}\right]\underset{\sim}{\phi}_{\tau\tau}=0 \tag{86}$$

It turns out that the elementary solutions to the acoustic potential equation are useful in identifying other elementary solutions to the aerodynamic potential equation. This is taken up in the next section. There are two ways in which one may obtain equation (86) from equation (85). However, the interpretation differs.

First, we note that in the derivation of equation (85), we linearized the potential about a uniform flow with velocity U in the positive x direction. (Or, what is the same, the body moves in the negative x direction.) If we assume the flow has zero velocity, then equation (85) takes the form of equation (86) and $\underset{\sim}{\phi}$ is identical to $\tilde{\phi}$ and so are the coordinate frames.

However, if we take a second approach, we see that equation (86) can be obtained from equation (85) by the follow. ıg simple translation (also known as the Gaussian transformation).

$$x_o = x - Ut \tag{87}$$

$$y_o = y \tag{88}$$

$$z_o = z \tag{89}$$

$$\tau = t \tag{90}$$

We see that the $(x_o y_o z_o)$ frame moves with velocity $U\hat{i}$ with respect to the (x, y, z) frame and is therefore motionless with respect to the undisturbed atmosphere. Next, we state that the potential in the (x, y, z) frame is the same as the potential in the (x_o, y_o, z_o) frame. We distinguish between the functional descriptions with an over-tilde and an under-tilde.

$$\tilde{\phi}(x, y, z, t) = \underset{\sim}{\phi}(x_o, y_o, z_o, \tau) \tag{91}$$

We use the chain rule and equations (87) through (90) to carry out the differentiation process. We denote differentiation with subscripts.

$$\tilde{\phi}_x = \underset{\sim}{\phi}_{x_o}(x_o)_x = \underset{\sim}{\phi}_{x_o} \tag{92}$$

$$\tilde{\phi}_t = \underset{\sim}{\phi}_{x_o}(x_o)_t + \underset{\sim}{\phi}_\tau \tau_t = -U\underset{\sim}{\phi}_{x_o} + \underset{\sim}{\phi}_\tau \tag{93}$$

We follow through with less detail for the higher derivatives using both the product rule and the chain rule for differentiation.

$$\tilde{\phi}_{xx} = \underset{\sim}{\phi}_{x_o x_o} \tag{94}$$

$$\tilde{\phi}_{xt} = -U\underset{\sim}{\phi}_{x_o x_o} + \underset{\sim}{\phi}_{x_o \tau} \tag{95}$$

$$\tilde{\phi}_{tt} = U^2 \underset{\sim}{\phi}_{x_o x_o} - 2U\underset{\sim}{\phi}_{x_o \tau} + \underset{\sim}{\phi}_{\tau\tau} \tag{96}$$

Of course, there is no change for derivatives in the lateral directions.

$$\tilde{\phi}_{yy} = \underset{\sim}{\phi}_{y_o y_o} \tag{97}$$

$$\tilde{\phi}_{zz} = \underset{\sim}{\phi}_{z_o z_o} \tag{98}$$

By substituting equations (94) through (98) into equation (85), we obtain equation (86).

The interpretation of the two approaches is different. The first approach is trivial. We simply set the velocity to zero. The second approach is a linear transformation. The aerodynamic potential equation (85) is identically the same as the acoustic potential equation in a uniformly moving frame.

SECTION VI

The Elementary Solution to the Acoustic Potential Equation

In the previous section, we showed how the acoustic potential equation (86) relates to the aerodynamic potential equation. In this section we seek an elementary solution to the acoustic potential equation. This is given in equation (109). We introduce the linear Laplace operator ∇^2 and restate the acoustic potential equation (86) as equation (99). (we drop the tilde underscore.)

$$\frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \tag{99}$$

In equation (99), we have not specified ϕ in any particular frame of reference. We now introduce the use of spherical coordinates where r is the radial measure, θ is the measure of longitude and λ is the measure of latitude.

$$\phi = \phi(r, \theta, \lambda, t) \tag{100}$$

The acoustic equation (99) takes the following form[1] in spherical coordinates.

$$\left[\frac{1}{a^2}\right]\frac{\partial^2\phi}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi}{\partial r}\right] + \left[\frac{1}{r^2\sin\lambda}\right]\frac{\partial}{\partial\lambda}\left[\sin\lambda\frac{\partial\phi}{\partial\lambda}\right] + \left[\frac{1}{r^2(\sin\lambda)^2}\right]\frac{\partial^2\phi}{\partial\theta^2} \tag{101}$$

We are looking for an elementary solution to the acoustic potential equation. This elementary solution may then be used in computing complex solutions by

1. Hildebrand, page (313).

the principle of superposition. More importantly, the elementary solution may be adapted to solve the aerodynamic potential equation (42). We make an educated guess and look for a solution to equation (101) which is spherically symmetric. We temporarily designate this solution with an overbar (not to be confused with the steady state solution in Section II).

$$\phi = \bar{\phi}(r, t) \tag{102}$$

A spherically symmetric flow is the same as a pulsating source (or sink) with radial streamlines. Physically, such a flow injects mass into the field. The spherically symmetric form of the acoustic equation is

$$\frac{\partial^2 \bar{\phi}}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \bar{\phi}}{\partial r}\right] \tag{103}$$

In order to maintain the undisturbed far field condition, we seek an elementary solution which dies off as $r \to \infty$. We choose the lowest order expression.

$$\bar{\phi}(r, t) = \frac{f(r, t)}{r} \tag{104}$$

By substituting equation (104) into equation (103), we obtain the following hyperbolic partial differential equation.

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial r^2} \tag{105}$$

Hyperbolic equations have characteristic solutions[1]. The two characteristic solutions to equation (105) are combined.

$$f(r, t) = f_i(r + at) + f_e(r - at) \tag{106}$$

1. see page 399 of Hildebrand

where f_i and f_e are any analytic function. Usually, the classical acoustic solution is placed in the following form which is obtained from equation (106) by modifying the form of f_i and f_e.

$$f(r,t) = g_i\left[t+\frac{r}{a}\right] + g_e\left[t-\frac{r}{a}\right] \tag{107}$$

One can easily verify that equation (107) is a solution of equation (105). Now, substituting equation (107) into equation (104), we obtain the result

$$\bar{\phi}(r,t) = \frac{g_i\left[t+\frac{r}{a}\right]}{r} + \frac{g_e\left[t-\frac{r}{a}\right]}{r} \tag{108}$$

If we plot g_e as a function of time (assuming some initial pulse) we see that g_e represents an expanding wave. Likewise, g_i represents an imploding wave. We choose $g_i = 0$ because the acoustic phenomena of interest here takes the form of an expanding wave from a central disturbance. We drop the subscript e and write

$$\bar{\phi}(r,t) = \frac{g\left[t-\frac{r}{a}\right]}{r} \tag{109}$$

The argument $(t-\frac{r}{a})$ is called retarded time because it accounts for the delay between the time the radiating disturbance was initiated at $r = 0$ until the time it reached the distance r. We may assume that the function $g(t)$ is known.

It is a straightforward matter to verify that equation (109) is a solution to the acoustic potential equation (86) given in the previous section in terms of cartesian coordinates. First we establish the following relations.

$$r = (x^2+y^2+z^2)^{1/2} \tag{110}$$

$$r_x = \frac{x}{r} \tag{111}$$

$$r_{xx} = \frac{1}{r} - \frac{x^2}{r^3} \tag{112}$$

We now proceed to develop expressions for the derivatives of $\phi = \bar{\phi}$ given in equation (109).

$$\bar{\phi}_x = \left[\frac{-x}{r^3}\right] g\left[t - \frac{r}{a}\right] - \left[\frac{x}{ar^2}\right] g'\left[t - \frac{r}{a}\right] \tag{113}$$

$$\bar{\phi}_{xx} = \left[\frac{3x^2}{r^5} - \frac{1}{r^3}\right] g\left[t - \frac{r}{a}\right] + \left[\frac{3x^2}{ar^4} - \frac{1}{ar^2}\right] g'\left[t - \frac{r}{a}\right] + \left[\frac{x^2}{a^2 r^3}\right] g''\left[t - \frac{r}{a}\right] \tag{114}$$

The derivatives with respect to y and z are derived similarly.

$$\bar{\phi}_{yy} = \left[\frac{3y^2}{r^5} - \frac{1}{r^3}\right] g\left[t - \frac{r}{a}\right] + \left[\frac{3y^2}{ar^4} - \frac{1}{ar^2}\right] g'\left[t - \frac{r}{a}\right] + \left[\frac{y^2}{a^2 r^3}\right] g''\left[t - \frac{r}{a}\right] \tag{115}$$

$$\bar{\phi}_{zz} = \left[\frac{3z^2}{r^5} - \frac{1}{r^3}\right] g\left[t - \frac{r}{a}\right] + \left[\frac{3z^2}{ar^4} - \frac{1}{ar^2}\right] g'\left[t - \frac{r}{a}\right] + \left[\frac{z^2}{a^2 r^3}\right] g''\left[t - \frac{r}{a}\right] \tag{116}$$

The derivative with respect to t is directly derived.

$$\bar{\phi}_{tt} = \frac{1}{r} g''\left[t - \frac{r}{a}\right] \tag{117}$$

We substitute equations (114), (115), (116) and (117) into the equation (86) and we see that it is satisfied. Thus, we have confirmed that $\phi = \bar{\phi}$ is an elementary solution to the acoustic potential equations.

Now we come to an important point. Up until now, we have assumed the point source is stationary. If the source is moving in time, relative to the point x_o, y_o, z_o, then the definition of r changes. From equation (110), we now have

$$r = [(x(t) - x_o)^2 + (y(t) - y_o)^2 + (z(t) - z_o)^2]^{1/2} \tag{118}$$

The expression for ϕ_{tt} in equation (117) is not correct for the moving source and the acoustic equation (86) is no longer satisfied. So we formulate a new elementary solution which represents a moving source. This is accomplished in the next section.

SECTION VII

The Moving Source

In this section, we obtain the elementary formula (145) for the moving source. In the next section, we transform this solution to the moving frame of reference. Then one may demonstrate that this formula does indeed satisfy the aerodynamic potential equation (42).

The following explanation is a review of the derivation provided by Garrick[1]. We superimpose a train of stationary sources, all in a line. These sources are pulsed in a sequence, thus giving the same effect as a single source moving at a constant speed. One may think of this as a motion picture film of a moving source derived from a series of photographic frames.

The pulse function $\delta(t)$ is defined to be 0 when $t \neq 0$ and to be 1 when $t = 0$. We can define the pulse function more eloquently with the following continuous function.

$$\delta(t) = \lim_{a \to 0} \begin{cases} 0 & t < -a \\ \frac{1}{2}\left[1 + \cos\left[\frac{\pi t}{a}\right]\right] & -a \leq t \leq a \\ 0 & t > a \end{cases} \tag{119}$$

The pulse function has the following effect when placed in the integrand. When we view the integration process as the limit of a series summation, the pulse

1. See page 671 of the work by Garrick

function selects only one value of the integrand and reduces the other series elements to zero.

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau \tag{120}$$

Given a set of orthogonal x_o, y_o and z_o axes, fixed to the atmosphere, place a series of stationary sources coincident with the x_o axis. This is shown in figure 1. These six sources shown are located at $x_o = \xi_i$ where the index i ranges from $1 \le i \le 6$.

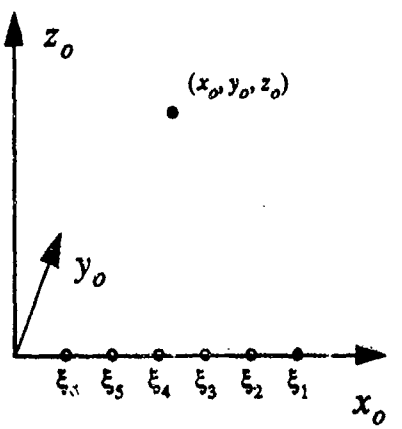

Figure 1. Acoustic Sources on the x_o Axis

All six stationary sources are assumed to act independently. Because the acoustic potential equation is linear, we can superimpose the elementary formula (109) relating the potential at (x_o, y_o, z_o) due to a source located at $x_o = \xi_i$ and $y_o = z_o = 0$.

$$\phi(x_o, y_o, z_o, t) = \sum_{i=1}^{6} \left[\frac{1}{r_i}\right] g\left[t - \frac{r_i}{a}\right] \tag{121}$$

where

$$r_i^2 = (x_o - \xi_i)^2 + (y_o)^2 + (z_o)^2 \tag{122}$$

If g is the pulse function, then equation (121) represents the potential which arises due to the six sources which are simultaneously pulsed at $t = 0$ with unit strength.

$$\phi(x_o, y_o, z_o, t) = \sum_{i=1}^{6} \left[\frac{1}{r_i}\right] \delta\left[t - \frac{r_i}{a}\right] \tag{123}$$

In equation (123), we see the effect of the retarded time. Even though all six sources are pulsed simultaneously, the effect at the point (x_o, y_o, z_o) is picked up (r_i/a) later. In other words, six simultaneous pulses are transmitted to the point (x_o, y_o, z_o) with a different delay.

Now, instead of pulsing simultaneously, we pulse the sources in a sequence, starting with the source at ξ_1 and ending with the source at ξ_6. Each source is pulsed with a time delay of $t = \tau_i$ relative to $t = 0$. Furthermore, rather than a unit strength, each source is pulsed with strength F_i. Equation (123) now takes the form

$$\phi(x_o, y_o, z_o, t) = \sum_{i=1}^{6} \left[\frac{1}{r_i}\right] (F_i)\, \delta\left[(t - \tau_i) - \frac{r_i}{a}\right] \tag{124}$$

Instead of six point sources, we may have many point sources along the x_o axis. In fact, we may define a continuum of sources as a limiting process as the number of point sources go to infinity. (While, this argument is not rigorously defended here, it can be shown that the desired formula relating the potential field due to a moving source does satisfy the aerodynamic potential equation.) The summation process of equation (124) is now treated as an integral

$$\phi(x_o, y_o, z_o, t) = \int_{-\infty}^{\infty} \left[\frac{1}{r(\tau)}\right] F(\tau)\, \delta\left[(t-\tau) - \frac{r(\tau)}{a}\right] d\tau \tag{125}$$

where τ is the dummy variable of integration which one may think of as running in parallel with time t. The variable τ points to some place in time for which a uniquely tagged point (or differential) source is active. We devise the definition of $r(\tau)$ based on r_i given in equation (122).

$$r^2(\tau) = (x_o - \xi_o(\tau))^2 + y_o^2 + z_o^2 \tag{126}$$

$F(\tau)\, d\tau$ is the differential strength of the source pulse which is uniquely identified by the time delay τ. Now, let the sources be pulsed at a uniform rate of $-U$ along the x axis. Furthermore, we specify when $\tau = 0$, there is a pulse at $x = 0$. In other words, we specify

$$\xi_o(\tau) = -U\tau \tag{127}$$

and equation (126) becomes

$$r^2(\tau) = (x_o + U\tau)^2 + y_o^2 + z_o^2 \tag{128}$$

We now strive to evaluate the integral equation (125). The resulting formula, equation (145), is the desired elementary solution for a moving source. In order to take direct advantage of the sifting property described by equation (120), we employ the following change of variable in equation (125) in order to simplify the process.

$$-\Theta = t - \tau - \frac{r(\tau)}{a} \tag{129}$$

Substituting for $r(\tau)$ from equation (128), we obtain

$$-\Theta = t - \tau - \left(\frac{1}{a}\right)\left((x_o + U\tau)^2 + y_o^2 + z_o^2\right)^{1/2} \tag{130}$$

We now modify equation (125)

$$\phi(x_o, y_o, z_o, t) = \int_{-\infty}^{\infty} \left[\frac{1}{r(\tau)}\right] F(\tau)\, \delta(\Theta) \left[\frac{d\tau}{d\Theta}\right] d\Theta \tag{131}$$

In the process of changing the variable of integration from τ to Θ, we require expressions for $\tau(\Theta)$ and $d\tau/d\Theta$. This task is easier than one may suppose at first. According to the sifting principle of equation (120), we evaluate the integrand at $\Theta = 0$ only. This gives the result

$$\phi(x_0, y_0, z_0, t) = \left[\left[\frac{1}{r(\tau)}\right] F(\tau) \left[\frac{d\tau}{d\Theta}\right]\right]_{\Theta=0} \tag{132}$$

We now evaluate equation (132) for the potential which arises from a moving source. This is achieved in equation (145) using equations (135) and (143).

From equation (130), obtain the following quadratic in τ for $\Theta = 0$.

$$(\beta^2)\tau^2 - 2\left[t + \frac{Ux_o}{a^2}\right]\tau + \left[t^2 - \frac{(x_o^2 + y_o^2 + z_o^2)}{a^2}\right] = 0 \tag{133}$$

where $\beta^2 = 1 - M^2$ and $M = U/a$ is the Mach number. From this quadratic equation, we expect two solutions for τ.

$$\tau = \left(\frac{1}{\beta^2}\right)\left[\left(t + \frac{Ux_o}{a^2}\right) \pm \frac{1}{a}\left((x_o + Ut)^2 + \beta^2 y_o^2 + \beta^2 z_o^2\right)^{1/2}\right] \tag{134}$$

For subsonic flow $(M < 1)$, we choose to limit $\tau < t$ which limits equation (134) to one root[1]. Equation (134) becomes

1. Garrick gives an excellent graphical explanation for the subsonic and supersonic cases in figures F.4a and F.4b on pages 674 and 676 respectively.

$$\tau = \left[\frac{1}{\beta^2}\right]\left[\left[t + \frac{Ux_o}{a^2}\right] - \left[\frac{\bar{R}}{a}\right]\right] \tag{135}$$

where

$$\bar{R} = \left((x_o + Ut)^2 + \beta^2 y_o^2 + \beta^2 z_o^2\right)^{1/2} \tag{136}$$

Equation (135) is the expression for τ when $\Theta = 0$ and $U < a$. We now compute $d\tau/d\Theta$ with $\Theta = 0$. Equation (130) is rearranged, squared and differentiated to obtain equation (137).

$$\frac{d}{d\Theta}\left[a^2(\tau - t - \Theta)^2\right] = \frac{d}{d\Theta}\left[(x_o + U\tau)^2 + y_o^2 + z_o^2\right] \tag{137}$$

Carry out the differentiation and then set $\Theta = 0$.

$$a^2(\tau - t)\left[\frac{d\tau}{d\Theta} - 1\right] = (x_o + U\tau)\,U\left[\frac{d\tau}{d\Theta}\right] \tag{138}$$

Now solve for $d\tau/d\Theta$.

$$\left[a^2(\tau - t) - U(x_o + U\tau)\right]\frac{d\tau}{d\Theta} = a^2(\tau - t) \tag{139}$$

Next, we observe from equation (129), for $\Theta = 0$

$$r = a(t - \tau) \tag{140}$$

Making the substitution in equation (139) gives

$$\left[-ar - U(x_o + U\tau)\right]\frac{d\tau}{d\Theta} = -ar \tag{141}$$

$$\frac{1}{r}\frac{d\tau}{d\Theta} = \frac{a}{ar + U(x_o + U\tau)} \tag{142}$$

Using equations (140) and (135), we substitute for r and τ on the right hand side of equation (142). After some simple algebraic manipulation, we arrive at the following simple result.

$$\frac{1}{r}\frac{d\tau}{d\Theta} = \frac{1}{\bar{\bar{R}}} \tag{143}$$

The definition for $\bar{R}$ was given in equation (136). Finally, we use equation (143) in equation (132) to obtain

$$\phi(x_o, y_o, z_o, t) = \left[\frac{1}{\bar{\bar{R}}}\right] F(\tau)\Big|_{\Theta=0} \tag{144}$$

From equation (135), we have an expression for τ when $\Theta = 0$. We make the substitution in equation (144).

$$\phi(x_o, y_o, z_o, t) = \left[\frac{1}{\bar{\bar{R}}}\right] F\left[\frac{1}{\beta^2}\left(t + \frac{Ux_o}{a^2} - \frac{\bar{R}}{a}\right)\right] \tag{145}$$

where $\bar{\bar{R}}$ is defined in equation (136).

This is the fundamental solution for the potential which arises due to a source moving along the x axis with constant velocity of $-U\hat{i}$. In the next section, we transform the coordinates to a moving frame.

SECTION VIII

The Elementary Solution to the Aerodynamic Potential Equation

Our objective of the past three sections has been to derive elementary solutions to the aerodynamic potential equation (42) which may be used to model the flow over wings and bodies. In Section V, we recognized that the aerodynamic potential equation is related to the acoustic potential equation by a simple Gaussian transformation. The coordinates axes of the acoustic potential equation are fixed to the atmosphere while the coordinate axes of the aerodynamic potential equation move with constant velocity $-U\hat{i}$ relative to the atmosphere. The elementary solution to the acoustic potential equation is a stationary point source with a spatial decay of $(1/r)$. We used a modified form of this solution (109) to obtain an elementary solution to the aerodynamic potential equation. Then a complication arose. We discovered that a simple translation of the stationary source in the x direction does not satisfy the acoustic potential equation. This mathematical complication is the result of compressibility (also referred to as the Doppler effect). Here, we are faced with the apparent compression of pressure wave fronts travelling upstream and the apparent expansion of pressure wave fronts travelling downstream. So, through the limiting process of superimposing a series of source pulses, we simulated a constant velocity source and derived the formula (145) for the resulting potential.

In this section, we apply a change of coordinates to the moving source solution (145) in the acoustic frame and thereby obtain the moving source solution (152) in the original constant velocity frame. This is the desired elementary solution to the aerodynamic potential equation. One may directly verify that equation (152) solves the aerodynamic potential equation by direct substitution.

Again, our objective is to mathematically model the flow over wings and bodies. One approach is to position a continuum of sources on the wing or body surface in order to disturb the flow and thereby satisfy the tangential flow boundary condition. This concept of a spatial continuum of sources will be discussed in the next section.

The moving coordinate frame is fixed relative to the structural geometry and has a velocity $-Ui$ relative to the acoustic coordinate frame. Therefore, a source moving with velocity $-U\hat{i}$ in the acoustic frame is now fixed relative to the moving frame. The potential which arises from a moving source was presented as equation (145). From equation (127) we know that at $t = 0$ the single moving source is located at the origin of the $(x_o y_o z_o)$ axes. We may modify the elementary solution (145) to model the potential $\underset{\sim}{\phi}(x_o y_o z_o)$ due to a single moving source which maintains a constant distance (ξ, η, ζ) relative to the source located along the x_o axis at $(\xi_o = -Ut)$. (We revert to the under-tilde to denote the potential in the stationary acoustic frame. Furthermore, the functional notation $\tilde{\underset{\sim}{\phi}}$ is used to denote the transitional state between the stationary and moving frames.)

$$\tilde{\underset{\sim}{\phi}}_s(x_o, y_o, z_o, \xi, \eta, \zeta, t) = \left[\frac{1}{\bar{R}}\right] F\left[\frac{1}{\beta^2}\left[t + \frac{U(x_o - \xi)}{a^2} - \frac{1}{a}\bar{R}\right]\right] \tag{146}$$

and from equation (136)

$$\bar{R} = \left((x_o + Ut - \xi)^2 + \beta^2 (y_o - \eta)^2 + \beta^2 (z_o - \zeta)^2\right)^{1/2} \tag{147}$$

Now we use equations (87) through (91) to change from the fixed $(x_o y_o z_o)$ coordinates to the moving (x, y, z) coordinates.

$$\tilde{\phi}_s(x, y, z, \xi, \eta, \zeta, t) = \left[\frac{1}{\bar{R}}\right] F\left[\frac{1}{\beta^2}\left[t + \frac{U(x - Ut - \xi)}{a^2} - \frac{1}{a}\bar{R}\right]\right] \tag{148}$$

and

$$\bar{R} = \left((x-\xi)^2 + \beta^2 (y-\eta)^2 + \beta^2 (z-\zeta)^2 \right)^{1/2} \tag{149}$$

We make algebraic simplifications to equation (148) to obtain the following equation.

$$\tilde{\phi}(x_o, y_o, z_o, \xi, \eta, \zeta, t) = \left[\frac{1}{\bar{R}}\right] F\left[t + \frac{M(x-\xi) - \bar{R}}{a\beta^2}\right] \tag{150}$$

As a final step, we give τ a new definition, not related to the dummy variable used in the previous section. Here, τ is the retarded variable and it represents the time delay incurred for a pulse to transit from its origin at (ξ, η, ζ) to the point (x, y, z).

$$\tau = \frac{-M(x-\xi) + \bar{R}}{a\beta^2} \tag{151}$$

The form of equation (150) is simplified.

$$\tilde{\phi}_s(x, y, z, \xi, \eta, \zeta, t) = \left[\frac{1}{\bar{R}}\right] F[t-\tau] \tag{152}$$

The subscript s is added to denote the source solution. Later a subscript d will denote the doublet solution. In the derivation of equations (152), (151) and (149) we closely followed the approach taken by Garrick. This is the fundamental moving point source solution to the aerodynamic potential equation. Thus, our interim objective has been achieved. Higher order solutions can (and will) be formulated by differentiating equation (152) with respect to x, y or z.

It can be shown by direct substitution that equation (152) solves the aerodynamic potential equation.

$$(1-M^2)\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz} - \left[\frac{2U}{a^2}\right]\tilde{\phi}_{xt} - \left[\frac{1}{a^2}\right]\tilde{\phi}_{tt} = 0 \tag{153}$$

SECTION IX

The Source Sheet

In the previous section, we showed that the fundamental source solution to the aerodynamic potential equation

$$(1-M^2)\,\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz} - \left[\frac{2U}{a^2}\right]\tilde{\phi}_{xt} - \left[\frac{1}{a^2}\right]\tilde{\phi}_{tt} = 0 \tag{154}$$

is given by the following simple formula.

$$\tilde{\phi}_s(x, y, z, \xi, \eta, \zeta, t) = \left[\frac{1}{\bar{R}}\right]F[t-\tau] \tag{155}$$

This is the formula for the potential at coordinates (x, y, z) due to a single point source at coordinates (ξ, η, ζ). The boundary condition for the flow over a thin wing was given in equation (79).

$$w = \frac{\partial \phi}{\partial z} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \tag{156}$$

where $h(x, y, t)$ describes the time dependent deformation of a thin wing in the (x, y) plane. The obvious question remains; how do we use equation (155) to solve for the flow over a wing? The answer is not simple and is the subject of the remainder of this text. We still need to formulate the source doublet in Section X and then we formulate the pressure doublet in Section XI. We use the concept of the doublet sheet to develop the integral formula in Section XII. In this section, we are introduced to the concept of a source sheet. While we will

not use the resulting formulae for a source sheet, the concept is directly applicable to a doublet sheet.

If we restrict our problem to the flow over a steady wing with no deflection (i.e. only the wing thickness is a consideration), the boundary condition equation (156) simplifies to

$$\frac{\partial \phi}{\partial z} = U \frac{\partial h}{\partial x} \tag{157}$$

Equation (155), for the elementary point source solution with temporally constant strength A, simplifies to

$$\tilde{\phi}_s(x, y, z, \xi, \eta, \zeta) = \left[\frac{A(\xi, \eta, \zeta)}{\bar{R}} \right] \tag{158}$$

Where A is the unknown strength of the source at the coordinates (ξ, η, ζ). In order to satisfy this boundary condition, we may superimpose n point sources located at $(x, y, z) = (\xi_i, \eta_i, 0)$. We satisfy the boundary condition on the wing surface[1] at n points located at (x_j, y_j, z_j). The thickness envelope is symmetric above and below the (x, y) plane so it is sufficient to satisfy the boundary condition on the top surface only. We differentiate equation (158) with respect to z and substitute the result into equation (157) for each static point source.

$$U \frac{\partial h_j}{\partial x} = \sum_{i=1}^{n} \left[\frac{\partial}{\partial z} \left[\frac{A(\xi_i, \eta_i)}{\bar{R}_i} \right] \right] \tag{159}$$

So we have a system of n equations and n unknowns which can be solved with linear algebra. One expects the accuracy of the solution to increase as the number of point sources and control points is increased. We can reformulate equation (159) as an integral if we consider the source in a limiting process.

1. By satisfying the boundary condition on the wing surface, we are inconsistent with our assumed linearization at the wing midplane. However, one expects the boundary condition on the wing surface to be more accurate than the midplane. Besides, we avoid problems with singularities.

$$U\frac{\partial h(x,y)}{\partial x} = \int_S \left[\frac{\partial}{\partial z}\left[\frac{A(\xi,\eta)}{\bar{R}}\right]\right] dS \tag{160}$$

Here, we have discretized the (x, y) plane into differential areas of size dS and located at $x = \xi$ and $y = \eta$. Each differential area has a source strength of AdS. Again, the steady function h is evaluated at (x, y). The radial measure $\bar{R}$ is now defined as

$$\bar{R} = \left[(x-\xi)^2 + \beta^2 (y-\eta)^2 + \beta^2 (z)^2\right]^{1/2} \tag{161}$$

If one is given the value of $h(x, y)$ at m points, equation (160) can be approximately solved for the unknown function $A(\xi, \eta)$ if $A(\xi, \eta)$ is defined in terms of m approximating components with constant coefficients. For instance, we may form a composite function, $A(\xi, \eta)$ by superimposing m polynomials in ξ and η, each polynomial weighted by a constant (but not yet specified) quantitiy. We integrate equation (160) for each polynomial. This results in a linear system of m equations with m unspecified constants. Alternatively, we may spatially discretize the wing planform and approximate $A(\xi, \eta)$ with a continuous spline function with m unspecified coefficients. Again, equation (160) is integrated to obtain a linear system of m equations with m unknown coefficients. We may think of this aerodynamic model as a linear system with m independent inputs ($h(x, y)$ at m points on the surface) and m dependent outputs (m polynomial constants). Once the approximating solution for $A(\xi, \eta)$ is obtained, the potential is determined using equation (158). Then we use the time invariant term of equation (52) to solve for the pressure.

$$P(x,y) = -\rho U\left[\frac{\partial \phi}{\partial x}\right] \tag{162}$$

Thus, one example of how one may use the elementary point source solution to obtain a continuous solution has been given. Keep in mind that we are solving the small disturbance problem and that the solution breaks down at stagnation conditions.

It turns out, that a single source sheet cannot generate a pressure difference across the (x, y) plane. Therefore, no lift can be generated. Mathematically, this is seen when one recognizes the symmetry of the potential above and below the (x, y) plane. For this reason, we investigate the source doublet in the next section.

SECTION X

The Source Doublet

In Section IV, the tangential flow boundary condition over a thin wing was separated into two linear components, wing thickness and wing deformation. Thus, we can treat the linear boundary value problem for a thin wing as two separate problems, the potential which arises due to the thickness envelope and the potential which arises with the deformation of the wing midplane. The total solution is the superposition of the two component solutions.

In the analysis of most linear systems, one considers the steady state condition and then superimposes the time dependent response. The steady state solution for a wing is a superposition of the pressure due to thickness and the pressure due to steady deformation of the midplane. The time dependent response is associated with the time dependent deformations of the wing midplane alone. This is an important consideration because we can mathematically model the flow over a deforming wing with an infinitely thin sheet.

In the previous section, we demonstrated a solution technique using a source sheet. However, it was pointed out that due to the symmetric nature of the source sheet (with respect to the (x, y) plane), it was not possible to develop a pressure differential. Therefore no lift can be generated with a single source sheet. However, it is possible to develop a pressure difference if two source sheets are placed in parallel. This can cause numerical problems if the two sheets are brought close together. This is not to say this has not been done. On the contrary, there are many examples where this is exactly what is done. However, for our linear analysis, this results in a waste of computational resources. Instead, we can formulate the limiting condition as two source sheets with opposing

strengths are brought close together. This is the source doublet sheet. In this Section, we describe the elementary point doublet formula.

Given the linear aerodynamic potential equation

$$\beta^2\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz} - \left[\frac{2U}{a^2}\right]\tilde{\phi}_{xt} - \left[\frac{1}{a^2}\right]\tilde{\phi}_{tt} = 0 \tag{163}$$

and the fundamental source solution of equation (152).

$$\phi_s = \frac{1}{R} f(t-\tau) \tag{164}$$

we show that the elementary solution

$$\phi_d = \frac{\partial}{\partial z}(\phi_s) \tag{165}$$

is also a solution. We substitute equation (165) into equation (163) to obtain

$$\beta^2\left[\frac{\partial}{\partial z}\phi_s\right]_{xx} + \left[\frac{\partial}{\partial z}\phi_s\right]_{yy} + \left[\frac{\partial}{\partial z}\phi_s\right]_{zz} - \left[\frac{2U}{a^2}\right]\left[\frac{\partial}{\partial z}\phi_s\right]_{xx} - \left[\frac{1}{a^2}\right]\left[\frac{\partial}{\partial z}\phi_s\right]_{tt} = 0 \tag{166}$$

Next, the order of differentiation is changed

$$\frac{\partial}{\partial z}\left[\beta^2(\phi_s)_{xx} + (\phi_s)_{yy} + (\phi_s)_{zz} - \left[\frac{2U}{a^2}\right](\phi_s)_{xt} - \left[\frac{1}{a^2}\right](\phi_s)_{tt}\right] = 0 \tag{167}$$

The term in the square brackets is known to be zero from equation (163). Equation (167) reduces to

$$\frac{\partial}{\partial z}[0] = 0 \tag{168}$$

Thus, we have shown that equation (165) is also a solution to the aerodynamic potential equation. We call this the source doublet or just the doublet solution. In order to give some physical significance, we investigate the source doublet for the steady incompressible case (set $M = 0$). Here we have the point source solution for a source of unit strength located at the origin. The potential is

$$\phi_s = \frac{1}{r} \tag{169}$$

with

$$r^2 = x^2 + y^2 + z^2 \tag{170}$$

According to equation (165), the potential which arises from a point doublet solution can be obtained by differentiation of the source solution with respect to z.

$$\phi_d = \frac{\partial \phi_s}{\partial z} = \frac{-z}{r^3} \tag{171}$$

Now, we take a second approach to arrive at equation (171). We bring two sources together from above and below the $z = 0$ plane as shown in figure (2). The strength of the sources are opposite and inversely proportional to the distance between them. Using equation (169) we obtain the combined potential

$$\phi_d = \lim_{\zeta \to 0} \left[\left[\frac{1}{2\zeta} \right] \left[\frac{1}{r_1} - \frac{1}{r_2} \right] \right] \tag{172}$$

Equation (170) is modified for each source as follows.

$$r_1^2 = x^2 + y^2 + (z + \zeta)^2 \tag{173}$$

$$r_2^2 = x^2 + y^2 + (z - \zeta)^2 \tag{174}$$

We temporarily assume $z \neq 0$. We can see that as $\zeta \to 0$, we have a zero in the numerator and a zero in the denominator.

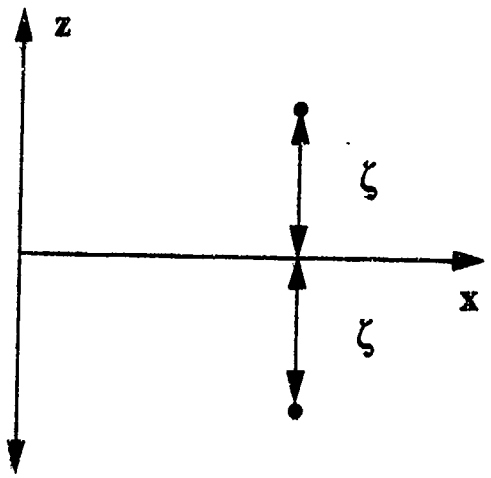

Figure 2. A Source Dipole

$$\phi_d = \lim_{\zeta \to 0}\left[\frac{\left[\frac{1}{r_1} - \frac{1}{r_2}\right]}{(2\zeta)}\right] \Rightarrow \frac{0}{0} \tag{175}$$

We use L'Hopital's Rule.

$$\phi_d = \lim_{\zeta \to 0}\left[\frac{\frac{\partial}{\partial\zeta}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]}{\frac{\partial}{\partial\zeta}(2\zeta)}\right] \tag{176}$$

$$\phi_d = \lim_{\zeta \to 0}\left[\frac{\frac{-(z+\zeta)}{r_1^3} - \frac{(z-\zeta)}{r_2^3}}{2}\right] \tag{177}$$

$$\phi_d = \frac{1}{2}\lim_{\zeta \to 0}\left[\frac{-z\,(r_1^3 + r_2^3)}{r_1^3 r_2^3}\right] + \frac{1}{2}\lim_{\zeta \to 0}\left[\frac{\zeta\,(r_2^3 - r_1^3)}{r_1^3 r_2^3}\right] \tag{178}$$

We now take the limit as $\zeta \to 0$ to obtain equation (179). Furthermore, we now allow z to go to zero. We see that equation (179) agrees with equation (171).

$$\phi_d = \frac{-z}{r^3} \tag{179}$$

So a source doublet is the limit as a source and sink (source with negative strength) are brought together with strengths inversely proportional to the distance between them. The same result is obtained by differentiation in equation (171).

In the same manner in which a source sheet was constructed by placing a point source in each differential area of a sheet, we can construct a doublet sheet by placing a point doublet solution in each differential area with the above doublet solution. However, there were restrictions placed on the above point doublet formula. It is limited to steady incompressible flow. In this report, doublets will be formulated for unsteady compressible flow.

For a point source, the potential which arises at any other point is proportional to $1/r$. Therefore, the potential field for a source solution is symmetric with respect to any plane passing through the point $r = 0$, including the (x, y) plane. It follows that the potential field arising from a source sheet in the (x, y) plane is symmetric with respect to the (x, y) plane. If the potential immediately above and below the (x, y) plane is identical, then it follows from equation (52) that the pressure immediately above and below the wing will be identical. Therefore, a single source sheet cannot generate a pressure difference.

For a point doublet, the potential is proportional to $-z/r^3$ and it follows that the potential field for a doublet sheet is antisymmetric with respect to the (x, y) plane. Since the potential is antisymmetric, we know from equation (52) that the

pressure above and below the (x, y) plane must also be antisymmetric. So, in contrast to the source sheet, we can develop a pressure difference with a single doublet sheet. This is a fundamentally simple but important concept.

SECTION XI

The Acceleration Potential

Up to this point, we have been using the velocity potential ϕ as the unknown variable in our linear aerodynamic system[1]. The input to our linear system is the wing deformation h. The output of our linear system is pressure p. However, our solution technique solves for the velocity potential first and pressure is computed later in a second step. In this section, we avoid the intermediate step of solving for the velocity potential and solve for pressure directly with the introduction of the pressure potential and subsequently the acceleration potential[2] ψ.

We start with the aerodynamic potential equation (42).

$$\beta^2\tilde{\phi}_{xx} + \tilde{\phi}_{yy} + \tilde{\phi}_{zz} - \left[\frac{2U}{a^2}\right]\tilde{\phi}_{xt} - \left[\frac{1}{a^2}\right]\tilde{\phi}_{tt} = 0 \tag{180}$$

Next, we differentiate with respect to t and then x to form the following equations

$$\beta^2(\tilde{\phi}_t)_{xx} + (\tilde{\phi}_t)_{yy} + (\tilde{\phi}_t)_{zz} - \left[\frac{2U}{a^2}\right](\tilde{\phi}_t)_{xt} - \left[\frac{1}{a^2}\right](\tilde{\phi}_t)_{tt} = 0 \tag{181}$$

$$\beta^2(\tilde{\phi}_x)_{xx} + (\tilde{\phi}_x)_{yy} + (\tilde{\phi}_x)_{zz} - \left[\frac{2U}{a^2}\right](\tilde{\phi}_x)_{xt} - \left[\frac{1}{a^2}\right](\tilde{\phi}_x)_{tt} = 0 \tag{182}$$

We multiply equation (181) by ρ_o and equation (182) by $\rho_o U$ and add them together to obtain equation (183).

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1. See Williams, Guderley and Lee for the non-singular formulation for the potential which arises from a doublet sheet.

2. ψ may also be described as a pressure doublet. To be consistent with equation (169), we should use the symbol p_d to describe a pressure doublet. However, we remain consistent with Vivian and Andrews and use ψ.

$$\beta^2 \rho_o (\tilde{\phi}_t + U\tilde{\phi}_x)_{xx} + \rho_o (\tilde{\phi}_t + U\tilde{\phi}_x)_{yy} + \rho_o (\tilde{\phi}_t + U\tilde{\phi}_x)_{zz} -$$

$$\left[\frac{2U}{a^2}\right] \rho_o (\tilde{\phi}_t + U\tilde{\phi}_x)_{xt} - \left[\frac{1}{a^2}\right] \rho_o (\tilde{\phi}_t + U\tilde{\phi}_x)_{tt} = 0 \qquad (183)$$

We recognize the following form of equation (52) within equation (183).

$$p = -\rho_o (\tilde{\phi}_t + U\tilde{\phi}_x) \qquad (184)$$

After making the substitution in equation (183), we arrive at the pressure potential equation

$$\beta^2 p_{xx} + p_{yy} + p_{zz} - \left[\frac{2U}{a^2}\right] p_{xt} - \left[\frac{1}{a^2}\right] p_{tt} = 0 \qquad (185)$$

Now the variable p seems to be doubly defined as both the pressure and the pressure potential. They mean the same thing. The form of equation (185) is mathematically identical to the form of the aerodynamic potential equation (180). Only the physical interpretation is different. Therefore, the elementary solutions to the aerodynamic potential equation (180) are also elementary solutions to the pressure potential equation (185). The elementary pressure source equation follows from the elementary potential source equation (152).

$$p_s = \left[\frac{1}{\bar{R}}\right] f(t - \tau) \qquad (186)$$

$$\tau = \frac{-M(x - \xi) + \bar{R}}{a\beta^2} \qquad (187)$$

$$\bar{R} = \left[(x - \xi)^2 + \beta^2 (y - \eta)^2 + \beta^2 (z - \zeta)^2 \right]^{1/2} \qquad (188)$$

We now restrict ourselves to harmonics in time. In other words, variable time dependency is replaced by a dependence on a constant frequency. By restricting the problem to constant frequencies ω, we obtain great computational savings. The computational cost of solving the aerodynamic flow over a body or wing in the time domain is great in comparison. So equation (186) for a pressure source p_s with strength A takes the complex form $p_s e^{i\omega t}$.

$$p_s = \left[\frac{A}{\bar{\bar{R}}}\right] exp(i\omega(t-\tau)) = \left[\frac{Ae^{-i\omega\tau}}{\bar{\bar{R}}}\right] exp(i\omega t) = \bar{p}_s e^{i\omega t} \qquad (189)$$

Now we use the formula (187) for retarded time τ in equation (189). For the modulus $\bar{p}_s$ on the right side of equation (189)

$$\bar{p}_s(x,y,z) = \frac{A}{\bar{\bar{R}}} e^{-i\omega\tau} = \frac{A}{\bar{\bar{R}}} exp\left[\frac{i\omega}{a\beta^2}(M(x-\xi)-\bar{R})\right] \qquad (190)$$

Now $\bar{p}_s$ is a symmetric function with respect to the $z = \zeta$ plane. This means that we cannot use equation (190) to model a pressure difference across a planar wing. We seek a formula for a pressure doublet. As with the source doublet, we differentiate equation (189) with respect to z to obtain the definition for the pressure doublet which will be called the acceleration potential ψ here. We divided by ρ_o in order to simplify the subsequent formulae and to bring these formulae in line with the original derivations[1].

$$\psi = \left[\frac{1}{\rho_o}\right]\frac{\partial}{\partial z}[p_s] = \left[\frac{1}{\rho_o}\right]\frac{\partial}{\partial z}[\bar{p}_s]\, e^{i\omega t} = \left[\frac{1}{\rho_o}\right]\bar{\psi}(x,y,z)\, e^{i\omega t} \qquad (191)$$

where the modulus of ψ is the differentiation of equation (190) with respect to z.

$$\bar{\psi}(x,y,z) = \frac{\partial}{\partial z}[\bar{p}_s] = \frac{\partial}{\partial z}\left[\frac{A}{\bar{\bar{R}}} exp\left[\frac{i\omega}{a\beta^2}(M(x-\xi)-\bar{R})\right]\right] \qquad (192)$$

1. See L. V. Andrews and H. T. Vivian.

Now $\psi = \bar{\psi} e^{i\omega t}$ is an elementary solution to equation (185) and $p = \rho_o \psi$ is the pressure which arises with the acceleration potential. Some useful relations are now derived. We define the non-dimensional pressure coefficient C_p and immediately specialize it to the acceleration potential.

$$C_p = \frac{2p}{\rho_o U^2} = \frac{2\psi}{U^2} \tag{193}$$

Next, we investigate the relationship between the velocity potential and the acceleration potential. This is important in deriving the boundary conditions for the pressure potential boundary value problem. We denote the harmonically oscillating potential as

$$\tilde{\phi}(x, y, z, t) = \bar{\phi}(x, y, z)\, e^{i\omega t} \tag{194}$$

The overvar indicates the complex modulus. The overbar here does not indicate the steady state condition as used in Section II. From $\psi = \frac{p}{\rho_0}$ and equation (184), we obtain[1]

$$\psi(x, y, z, t) = -\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right](\bar{\phi}(x, y, z)\, e^{i\omega t}) \tag{195}$$

Carrying out the operations

$$\bar{\psi}(x, y, z) = -U\bar{\phi}_x(x, y, z) - i\omega\bar{\phi}(x, y, z) \tag{196}$$

We arrive at the inverse relation to equation (196) by using $p = \left[\frac{\bar{\psi}}{\rho_0}\right] e^{i\omega t}$ in equation (73).

$$\bar{\phi}(x, y, z) = \frac{-1}{U} exp\left[\frac{-i\omega x}{U}\right] \int_{-\infty}^{x} exp\left[\frac{i\omega\lambda}{U}\right] \bar{w}(\lambda, y, z)\, d\lambda \tag{197}$$

1. Here, we can see how the term "acceleration potential" arises. It is the total derivative of the velocity potential. The total derivative is a derivative with respect to time relative to a steady translating frame of reference. The order of differentiation is optional. Therefore, the acceleration field is related to the acceleration potential by the gradient.

SECTION XII

The Integral Formula

In the previous section, we identified the elementary solutions to equation (185). We derived the following elementary pressure doublet expression (192), otherwise known as the acceleration potential.

$$\bar{\psi}(x,y,z) = (A)\frac{\partial}{\partial z}\left[\frac{1}{\bar{R}}exp\left[\frac{i\omega}{a\beta^2}(M(x-\xi)-\bar{R})\right]\right] \tag{198}$$

where $A = A(\omega)$ is the amplitude of the oscillations at any given frequency. Also, we have

$$\bar{R} = \left[(x-\xi)^2+\beta^2(y-\eta)^2+\beta^2(z-\zeta)^2\right]^{1/2} \tag{199}$$

Equation (198) gives the pressure field which arises from a single pressure doublet. In this section, we extend the single pressure doublet to a doublet sheet. As indicated in Section X for the source doublet (velocity potential), the pressure doublet sheet is also suitable for modelling the pressure difference between the upper and lower surfaces of a thin wing. Our goal in this section is to develop the integral equation[1] (224) which describes the upwash generated by a pressure doublet sheet. (For our aerodynamic problem of flow over a wing, the upwash $\bar{w}$ is a known function. In later sections of this report, we will see how to carry out the integral of equation (224) with unknown pressure Δp.) First we

1. L. V. Andrews and H. T. Vivian, pp 15-20.

develop the relationship between the amplitude A and the pressure difference ΔP across a pressure doublet sheet. This is given in equation (215).

The first step is to carry out the derivative with respect to z in equation (198). As a preliminary step we evaluate the following.

$$\frac{\partial}{\partial z}\bar{R} = \frac{\beta^2 (z-\zeta)}{\bar{R}} \tag{200}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\bar{R}}\right) = \frac{-\beta^2 (z-\zeta)}{\bar{R}^3} \tag{201}$$

Now, we carry through with the derivative in equation (198).

$$\bar{\psi}(x, y, z) = A\beta^2 (z-\zeta)\left[\frac{1}{\bar{R}^2} - \frac{1}{\bar{R}^3}\right] exp\left[\frac{i\omega}{a\beta^2}(M(x-\xi) - \bar{R})\right] \tag{202}$$

This is the formula for a single pressure doublet. This formula describes the pressure ψ at coordinates (x, y, z) due to a pressure doublet at coordinates (ξ, η, ζ). We now consider a continuum of doublets in the $(\xi, \eta, \zeta = 0)$ plane. Each differential area dS is given a doublet strength of AdS. While the choice of differential partitioning is really somewhat arbitrary, we will indicate a differential area as a rectangle of area $dS = d\xi d\eta$ for the time being.

$$\bar{\psi}(x, y, z) = \beta^2 z \int_S A(\xi, \eta)\left[\frac{1}{\bar{R}^2} - \frac{1}{\bar{R}^3}\right] exp\left[\frac{i\omega}{a\beta^2}(M(x-\xi) - \bar{R})\right] d\xi d\eta \tag{203}$$

The next task is to determine what value $\bar{\psi}$ takes as we approach the surface. That is, we evaluate

$$\lim_{z \to 0} \bar{\psi}(x, y, z) \tag{204}$$

The presence of the $\bar{R}^3$ and $\bar{R}^2$ terms in the denominator of equation (203) makes the integrand singular to order O^{-3} and O^{-2} when $z = 0$ and when the

coordinates (x, y) lie within the domain of S. This can be treated in the following manner.

Construct a small circle with radius ρ_o around the point $(x, y, z = 0)$ in the plane of the wing. First, we integrate over the region outside of the circle and clearly, in the limit as $z \to 0$, the contribution to $\bar{\psi}$ goes to zero. Only the portion of the doublet surface within the radius ρ_o contributes to ψ. Now, we change coordinates such that

$$(x-\xi) = \rho\cos(\theta)$$

$$(y-\eta) = \rho\sin(\theta)$$

$$d\xi d\eta = \frac{1}{\beta}\rho d\rho d\theta \quad .$$

For ρ and z sufficiently small, we can assume

$$exp\left(\frac{i\omega}{a\beta^2}(M(x-\xi)-\bar{R})\right) \approx \lim_{\varepsilon\to 0}(\cos\varepsilon + i\sin\varepsilon) \approx 1 \qquad (205)$$

Furthermore, we make $z/\bar{R}^3$ significantly bigger than $z/\bar{R}^2$ by choosing ρ_o and z to be small. Finally, we assume $A(\xi, \eta)$ is constant within our small circle. So we evaluate

$$\lim_{z\to 0}\bar{\psi}(x, y, z) = \lim_{z\to 0} A\int_0^{2\pi}\int_0^{\rho_o}\left[\frac{-\beta z\rho}{(\beta^2 z^2+\rho^2)^{3/2}}\right]d\rho d\theta \qquad (206)$$

$$\lim_{z\to 0}\bar{\psi}(x, y, z) = \lim_{z\to 0} A\int_0^{\rho_o}\left[\frac{-2\pi\beta z\rho}{(\beta^2 z^2+\rho^2)^{3/2}}\right]d\rho \qquad (207)$$

$$\lim_{z\to 0}\bar{\psi}(x,y,z) = \lim_{z\to 0}(2\pi A)\int_0^{\rho_o}\left[\frac{-\rho}{\beta^2 z^2\left(1+\left[\frac{\rho}{\beta z}\right]^2\right)^{3/2}}\right]d\rho \tag{208}$$

We now employ the substitution $\rho = \beta z\bar{\rho}$.

$$\lim_{z\to 0}\bar{\psi}(x,y,z) = \lim_{z\to 0}(2\pi A)\int_0^{\frac{\rho_o}{\beta z}}\left[\frac{-\bar{\rho}}{(1+\bar{\rho}^2)^{3/2}}\right]d\bar{\rho} \tag{209}$$

$$\lim_{z\to 0}\bar{\psi}(x,y,z) = \lim_{z\to 0}(2\pi A)\,(\bar{\rho}^2+1)^{-1/2}\Big|_{\bar{\rho}_o=0}^{\bar{\rho}_o=\frac{\rho_o}{\beta z}} \tag{210}$$

We end with the simple result that as one approaches the doublet sheet from the top side that the acceleration potential

$$\lim_{z\to 0}\bar{\psi}(x,y,z) = -2\pi A \tag{211}$$

We get a similar result for the case where the doublet sheet is approached from the opposite side.

$$\lim_{-z\to 0}\bar{\psi}(x,y,z) = 2\pi A \tag{212}$$

The jump in $\bar{\psi}$ across the doublet sheet, going from top to bottom is

$$\Delta\bar{\psi}(x,y) = 4\pi A \tag{213}$$

We recall that the relation between the acceleration potential and pressure is proportional to the density, such that $p = \psi/\rho$. So we see from equation (213), the pressure jump modulus across the doublet sheet is simply

$$\Delta\bar{p}(x,y) = 4\pi\rho A \tag{214}$$

Our convention is such that positive $\Delta\bar{p}$ results in positive lift with negative pressure on the positive z side of the wing and positive pressure on the negative z side of the wing. So we have the following expression for A in terms of $\Delta\bar{p}$.

$$A = \frac{\Delta\bar{p}}{4\pi\rho} \tag{215}$$

We make the substitution into equation (203) and obtain the following result.

$$\bar{\Psi}(x,y,z) =$$

$$\frac{\beta^2 z}{4\pi\rho}\int_S \Delta\bar{p}(\xi,\eta)\left[\frac{1}{\bar{R}^2}-\frac{1}{\bar{R}^3}\right]exp\left[\frac{i\omega}{a\beta^2}(M(x-\xi)-\bar{R})\right]d\xi d\eta \tag{216}$$

We desire an explicitly linear relationship between the pressure jump across a doublet sheet and the linear boundary condition, which is the normal component of flow. Our attention was diverted in order to obtain equation (215). We will use this relation later in our effort to obtain the final integral formula. So we reorient our attention and begin our development of this formulation with equation (197) of the last section. For a single oscillating doublet located at $x = \xi$, we have an oscillating potential of

$$\bar{\phi}(x,y,z) = \frac{-1}{U}exp\left[\frac{-i\omega(x-\xi)}{U}\right]\int_{-\infty}^{x-\xi} exp\left[\frac{i\omega\lambda}{U}\right]\bar{\Psi}(\lambda,y,z)\,d\lambda \tag{217}$$

We substitute for $\bar{\Psi}$ with the expression found in equation (198).

$$\bar{\phi}(x,y,z) = \frac{-A}{U}exp\left[\frac{-i\omega(x-\xi)}{U}\right] \times$$

$$\int_{-\infty}^{x-\xi} exp\left[\frac{i\omega\lambda}{U}\right]\left(\frac{\partial}{\partial z}\left[\frac{1}{\bar{R}}exp\left[\frac{i\omega}{a\beta^2}(M(\lambda-\xi)-\bar{R})\right]\right]\right)d\lambda \tag{218}$$

We can move the derivative with respect to z from under the integral and we can combine exponential terms. Equation (218) becomes

$$\bar{\phi}(x, y, z) =$$

$$\left[\frac{-A}{U}\right]\frac{\partial}{\partial z}\left[exp\left[\frac{-i\omega(x-\xi)}{U}\right]\int_{-\infty}^{x-\xi}\left(\frac{1}{\bar{R}}exp\left[i\omega\left(\frac{\lambda}{U}+\frac{M\lambda}{a\beta^2}-\frac{\bar{R}}{a\beta^2}\right)\right]\right)d\lambda\right] \quad (219)$$

The z component of velocity is related to the velocity potential as

$$\bar{w}(x, y, z) = \frac{\partial}{\partial z}(\bar{\phi}) \quad (220)$$

Thus, we compute $\bar{w}(x, y, z)$ from equation (219).

$$\bar{w}(x, y, z) =$$

$$\left[\frac{-A}{U}\right]\frac{\partial^2}{\partial z^2}\left[exp\left[\frac{-i\omega(x-\xi)}{U}\right]\int_{-\infty}^{x-\xi}\frac{1}{\bar{R}}exp\left[i\omega\left(\frac{\lambda}{U}+\frac{M\lambda}{a\beta^2}-\frac{\bar{R}}{a\beta^2}\right)\right]d\lambda\right] \quad (221)$$

We now consider a continuous sheet of doublets in the identical sense as was introduced in equation (203). Then substitute for A using equation (215).

$$\bar{w}(x, y, z) = \left[\frac{-1}{4\pi\rho U}\right]\iint_S(\Delta\bar{p})\,exp\left[\frac{-i\omega(x-\xi)}{U}\right]\times$$

$$\frac{\partial^2}{\partial z^2}\left[\int_{-\infty}^{x-\xi}\frac{1}{\bar{R}}exp\left[i\omega\left(\frac{\lambda}{U}+\frac{M\lambda}{a\beta^2}-\frac{\bar{R}}{a\beta^2}\right)\right]d\lambda\right]d\xi d\eta \quad (222)$$

This expression can be condensed slightly[1]. Again, equation (223) is the formula for the upwash $\bar{w}$ generated by a harmonically oscillating doublet sheet in the $z = 0$ plane.

1. Thereby putting equation (222) in the identical form as in the reference of Andrew and Vivian.

$$\bar{w}(x,y,z) = \left[\frac{-1}{4\pi\rho U}\right]\iint_S (\Delta\bar{p})\, exp\left[\frac{-i\omega(x-\xi)}{U}\right]$$
$$\frac{\partial^2}{\partial z^2}\left[\int_{-\infty}^{x-\xi}\frac{1}{\bar{R}}exp\left[\frac{i\omega}{U\beta^2}(\lambda - M\bar{R})\right]d\lambda\right]d\xi d\eta \qquad (223)$$

We choose to abbreviate this equation as follows.

$$\bar{w}(x,y,z) = \left[\frac{-1}{4\pi\rho U}\right]\iint_S \Delta\bar{p}K((x-\xi),(y-\eta),z)\,d\xi d\eta \qquad (224)$$

where

$$K(x_o,y_o,z_o) = exp\left(\frac{-i\omega x_o}{U}\right)\frac{\partial^2}{\partial z^2}\left[\int_{-\infty}^{x_o}\frac{1}{\bar{R}}exp\left[\frac{i\omega}{U\beta^2}(\lambda - M\bar{R})\right]d\lambda\right] \qquad (225)$$

and we essentially repeat equation (199).

$$\bar{R} = (\lambda^2 + \beta^2 y_o^2 + \beta^2 z_o^2)^{1/2} \qquad (226)$$

$K(x_o,y_o,z_o)$ is known as the Kernel function and is the topic of the next section.

For the purpose of this report, we have restricted ourselves to a single planar wing. As such, we may be tempted to immediately take the limit as $z \to 0$ in equation (224). However, we still need to evaluate the derivatives with respect to z in equation (225).

SECTION XIII

The Kernel Function

In the last section, we derived the integral equation (224) which relates the modulus of the unknown pressure difference $\Delta\bar{p}\,(x, y)$ to the modulus of the z component of flow $\bar{w}\,(x, y, z)$. We know the z component of flow at the wing surface (boundary condition) where $z = 0$. This will manifest singular behavior which will be addressed in the following section. In this section, we evaluate the Kernal function.

$$K\,(x_o, y_o, z_o) = exp\left[\frac{-i\omega x_o}{U}\right]\frac{\partial^2}{\partial z^2}\left[\int_{-\infty}^{x_o}\frac{1}{\bar{R}}exp\left[\frac{i\omega}{U\beta^2}\,(\lambda - M\bar{R})\right]d\lambda\right] \qquad (227)$$

The objective of this section is to carry out the differentiation with respect to z in equation (227). The resulting formula is given in equation (257). We isolate the integral expression in equation (227) and label it as I_o.

$$K\,(x_o, y_o, z_o) = exp\left[\frac{-i\omega x_o}{U}\right]\frac{\partial^2}{\partial z^2}\,[I_o] \qquad (228)$$

$$I_o = \int_{-\infty}^{x_o}\frac{1}{\bar{R}}exp\left[\frac{i\omega}{U\beta^2}\,(\lambda - M\bar{R})\right]d\lambda \qquad (229)$$

We define two new variables r_1 and k_1 which will be used throughout the remainder of this section.

$$r_1 = (y_o^2 + z_o^2)^{1/2} \tag{230}$$

$$k_1 = \frac{\omega r_1}{U} \tag{231}$$

This results in the following formula for $\bar{R}$

$$\bar{R} = (\lambda^2 + \beta^2 r_1^2)^{1/2} \tag{232}$$

In the evaluation of equation (229), we use a variable substitution. Let

$$\nu = \frac{\lambda}{\beta r_1} \tag{233}$$

$$I_o = \int_{-\infty}^{\nu_1} \frac{1}{(\nu^2+1)^{1/2}} exp\left[\left(\frac{i}{\beta}\right) (\nu k_1 - M k_1 (\nu^2+1)^{1/2}) \right] d\nu \tag{234}$$

where

$$\nu_1 = \left[\frac{\omega}{\beta U}\right] \frac{x_o}{k_1} \tag{235}$$

Now use another variable substitution. We define a new variable u.

$$u = \frac{-1}{\beta}\left[\nu - M(1+\nu^2)^{1/2}\right] \tag{236}$$

We obtain from equation (234)

$$I_o = \int_{u_1}^{\infty} \left[\frac{exp(-ik_1 u)}{(1+u^2)^{1/2}} \right] du \tag{237}$$

where

$$u_1 = \frac{-x_o + M\bar{R}}{r_1\beta^2} \tag{238}$$

Equation (237) is used in conjunction with equation (228). In the course of evaluating the derivative of equation (228), we use the chain rule

$$\frac{\partial}{\partial z} = \frac{\partial r_1}{\partial z}\frac{\partial}{\partial r_1} = \frac{z}{r_1}\frac{\partial}{\partial r_1}. \tag{239}$$

and we obtain

$$K(x_o, y_o, z_o) = exp\left(\frac{-i\omega x_o}{U}\right)\left[\left(\frac{1}{r_1} - \frac{z_o^2}{r_1^3}\right)\frac{\partial I_o}{\partial r_1} + \left(\frac{z_o^2}{r_1^2}\right)\frac{\partial^2 I_o}{\partial r_1^2}\right] \tag{240}$$

This can be put in a convenient form proposed by Landahl[1]. We will eventually restrict our formula to modelling the flow over planar surfaces in the (x, y) plane. Landahl's equation represents a more general case for any planar surface parallel to the x axis. We follow his development and then let z_o go to zero. Equation (240) is equivalent to the following equation (241).

$$K(x_o, y_o z_o) = exp\left[\frac{-i\omega x_o}{U}\right]\left[\frac{1}{r_1}\frac{\partial I_o}{\partial r_1} + \left[\frac{z_o^2}{r_1}\right]\frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\frac{\partial I_o}{\partial r_1}\right]\right] \tag{241}$$

Again, the expression for $\tilde{I}_o$ is given in equation (237). Now we use Landahl's approach and make the following substitution of variable in equation (237).

$$t = ur_1 \tag{242}$$

$$I_o = \int_{t_1}^{\infty}\left[\frac{exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2 + t^2)^{1/2}}\right]dt \tag{243}$$

1. See the article by Landahl

where

$$t_1 = u_1 r_1 = \frac{1}{\beta^2}(M\bar{R} - x_o) \tag{244}$$

In order to evaluate the derivatives of I_o in equation (241), we use Leibnitz's Rule[1] and equation (243).

$$\frac{\partial}{\partial r_1}(I_o) = \int_{t_1}^{\infty} \frac{\partial}{\partial r_1}\left(\frac{exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2 + t^2)^{1/2}}\right) dt - \left(\frac{exp\left[\frac{-i\omega t_1}{U}\right]}{(r_1^2 + t_1^2)^{1/2}}\right)\left[\frac{\partial t_1}{\partial r_1}\right] \tag{245}$$

Consider equation (245). From equations (232) and (244) we have

$$\left[\frac{\partial t_1}{\partial r_1}\right] = \frac{M}{\beta^2}\left[\frac{\partial \bar{R}}{\partial r_1}\right] = \frac{M}{\beta^2}\left[\frac{\beta^2 r_1}{\bar{R}}\right] = \frac{M r_1}{\bar{R}} \tag{246}$$

Next, we carry out the derivative in the integrand and equation (245) becomes

$$\frac{\partial}{\partial r_1}(I_o) = \int_{t_1}^{\infty} \left(\frac{-r_1 exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2 + t^2)^{3/2}}\right) dt - \left(\frac{exp\left[\frac{-i\omega t_1}{U}\right]}{(r_1^2 + t_1^2)^{1/2}}\right)\left[\frac{M r_1}{\bar{R}}\right] \tag{247}$$

Next, change the variable of integration back from t to u using equation (242). Equation (247) becomes

$$\frac{\partial}{\partial r_1}(I_o) = \left[\frac{-1}{r_1}\right] \int_{u_1}^{\infty} \left[\frac{exp(-ik_1 u)}{(1+u^2)^{3/2}}\right] du - \left[\frac{M}{\bar{R}}\right]\left[\frac{exp(-ik_1 u_1)}{(1+u^2)^{1/2}}\right] \tag{248}$$

Next, we evaluate the rest of equation (241). Starting with equation (247),

1. See the footnote on page 14 of this text

$$\frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\frac{\partial I_o}{\partial r_1}\right] = \frac{\partial}{\partial r_1}\left[\int_{t_1}^{\infty}\left(\frac{-exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2+t^2)^{3/2}}\right)dt - \left[\frac{M}{\overline{R}}\right]\left(\frac{exp\left[\frac{-i\omega t_1}{U}\right]}{(r_1^2+t_1^2)^{1/2}}\right)\right] \tag{249}$$

Due to the complexity of the formulation, we partition the right hand side of equation (249).

$$\frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\frac{\partial I_o}{\partial r_1}\right] = H_1 + H_2 \tag{250}$$

H_1 and H_2 are defined in equations (251) and (254).

$$H_1 = \frac{\partial}{\partial r_1}\int_{t_1}^{\infty}\left(\frac{-exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2+t^2)^{3/2}}\right)dt \tag{251}$$

Now, use Leibnitz's rule in equation (251).

$$H_1 = \int_{t_1}^{\infty}\left(\frac{3r_1 exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2+t^2)^{5/2}}\right)dt + \left(\frac{exp\left[\frac{-i\omega t}{U}\right]}{(r_1^2+t_1^2)^{3/2}}\right)\left[\frac{\partial t_1}{\partial r_1}\right] \tag{252}$$

We get the following result using the relationship between t to u in equation (242).

$$H_1 = \left(\frac{3}{r^3}\right)\int_{u_1}^{\infty}\left(\frac{exp\,(-ik_1 u)}{(1+u^2)^{5/2}}\right)du + \left(\frac{M}{r_1^2\overline{R}}\right)\left(\frac{exp\,(-ik_1 u)}{(1+u^2)^{3/2}}\right) \tag{253}$$

Next, we evaluate H_2 in equation (250).

$$H_2 = \frac{\partial}{\partial r_1}\left[\left[\frac{-M}{\bar{R}}\right]\left(\frac{exp\left[\frac{-i\omega t_1}{U}\right]}{(r_1^2+t_1^2)^{1/2}}\right)\right] \tag{254}$$

We carry out the differentiation operation.

$$H_2 = \left[\frac{i\omega}{U}\right]\left[\frac{\partial t_1}{\partial r_1}\right]\left[\frac{M}{\bar{R}}\right]\left[\frac{exp\,(\frac{-i\omega t_1}{U})}{(r_1^2+t_1^2)^{1/2}}\right] +$$

$$\left[\frac{M}{\bar{R}}\right]\left[\frac{exp\,(\frac{-i\omega t_1}{U})}{(r_1^2+t_1^2)^{3/2}}\right]\left[r_1+t_1(\frac{\partial t_1}{\partial r_1})\right] +$$

$$\left(\frac{exp\,(\frac{-i\omega t_1}{U})}{(r_1^2+t_1^2)^{1/2}}\right)\left(\frac{M}{\bar{R}^2}\right)\left[\frac{\partial \bar{R}}{\partial r_1}\right] \tag{255}$$

Now use equation (246) in equation (255). After some basic algebraic operations the following result is obtained.

$$H_2 = \left[\left(\frac{ik_1M^2}{r_1\bar{R}^2}\right)+\left(\frac{M\beta^2}{\bar{R}^3}\right)\right]\left[\frac{exp\,(-ik_1u_1)}{(1+u_1^2)^{1/2}}\right] + \tag{256}$$

$$\left[\left(\frac{M}{r_1^2\bar{R}}\right)+\left(\frac{M^2u_1}{r_1\bar{R}^2}\right)\right]\left[\frac{exp\,(-ik_1u_1)}{(1+u_1^2)^{3/2}}\right]$$

We now have an algebraic expression for the Kernel function given in equation (241) using equations (248) and (250). Equations (253) and (256) are used in place of equation (250). It is left to the reader to consolidate these equations. The operation is simple; it is not practical to display such a lengthy formula in this text. This completes our differentiation with respect to z in equation (?27).

Landahl has provided a compact form of our lengthy formula. At this point, we switch over to Landahl's[1] notation in order to follow his work closely. We will use the expanded terms following equation (241). Equation (241) can be placed in the form

$$K(x_o, y_o, z_o) = exp\left[\frac{-i\omega x_o}{U}\right]\frac{[K_1T_1 + K_2T_2]}{r_1^2} \tag{257}$$

where the terms are directly relatable to the two terms of equation (241).

$$K_1 = r_1\left[\frac{\partial I_o}{\partial r_1}\right] \tag{258}$$

$$K_2 = r_1^3\left[\frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\frac{\partial I_o}{\partial r_1}\right]\right] \tag{259}$$

In our development, we have limited ourselves to a doublet sheet in the (x, y) plane. The formula given by Landahl is more general, representing any doublet sheet which is parallel to the x axis. We make the appropriate modification to the expressions for T_1 and T_2 by setting $\gamma(s) = 0$ and $\gamma(\sigma) = 0$. Thus T_1 and T_2 become

$$T_1 = 1 \tag{260}$$

1. Take note that in the report by Landahl, there is an error in the K_2 term. This is clear when compared to equation (241) in this section.

$$T_2 = \left[\frac{z_o}{r_1}\right]^2 \tag{261}$$

We use equation (248) in Landahl's expression for K_1 and we use equation (250) in his expression for K_2. We obtain Landahl's resulting[1] formula for K_1 and K_2.

$$K_1 = -I_1 - \left[\frac{Mr_1}{\bar{R}}\right]\left[\frac{exp\,(-ik_1u_1)}{(1+u_1^2)^{1/2}}\right] \tag{262}$$

$$K_2 = 3I_2 + \left[\frac{ik_1M^2r_1^2}{\bar{R}^2}\right]\left[\frac{exp\,(-ik_1u_1)}{(1+u_1^2)^{1/2}}\right] +$$

$$\left[\frac{Mr_1}{\bar{R}}\right]\left[(1+u_1^2)\left(\frac{\beta^2r_1^2}{\bar{R}^2}\right) + (2) + \left[\frac{Mr_1u_1}{\bar{R}}\right]\right]\left[\frac{exp\,(-ik_1u_1)}{(1+u_1^2)^{3/2}}\right] \tag{263}$$

where

$$I_1 = \int_{u_1}^{\infty}\left(\frac{exp\,(-ik_1u)}{(1+u^2)^{3/2}}\right)du \tag{264}$$

and

$$I_2 = \int_{u_1}^{\infty}\left(\frac{exp\,(-ik_1u)}{(1+u^2)^{5/2}}\right)du \tag{265}$$

Differentiation with respect to z within the kernel function equation (227) is complete. The resulting equation for the kernel function is given by equation

1. The difference in the minus sign between this result and Landahl's result is accounted for in the minus sign added in equation (224).

(257). We might feel free to take the limit as z goes to zero for our planar wing. In equation (257) it turns out that, because of the singular nature at $z = 0$, this is not entirely recommended. However, we note that

$$\lim_{z_o \to 0} T_2 = 0$$

except when $y_o = 0$ in which case

$$\lim_{z_o \to 0} T_2 = 1.$$

We also note that K_2 is finite everywhere. When $K_2 T_2$ is added to $K_1 T_1$ in equation (257) and K is integrated in equation (224), the contribution of $K_2 T_2$ is zero when z_o is zero. Therefore, for a planar wing, we can immediately state,

$$\lim_{z_o \to 0} K(x_o, y_o, z_o) = \left(\frac{K_1}{r_1^2}\right) exp\left[\frac{-i\omega x_o}{U}\right] \tag{266}$$

Equation (264) for I_1, can be modified for improved computation. We assume[1] $u_1 \geq 0$.

$$I_1 = \int_{u_1}^{\infty} \frac{exp(-ik_1 u)}{(1+u^2)^{3/2}} du = \int_{u_1}^{\infty} [exp(-ik_1 u)] \left[\frac{du}{(1+u^2)^{3/2}}\right] \tag{267}$$

Integrate by parts to obtain

$$I_1 = \left[exp(-ik_1 u) \left(\frac{u}{(1+u^2)^{1/2}}\right)\right]_{u_1}^{\infty} +$$

$$ik \int_{u_1}^{\infty} \left(\frac{u}{(1+u^2)^{1/2}}\right) exp(-ik_1 u)\, du \tag{268}$$

1. Use equation (275) for $u_1 < 0$.

Evaluate the term in square brackets.

$$I_1 = \left[exp\,(-ik_1\infty) - exp\,(-ik_1u_1)\,\frac{u_1}{(1+u_1^2)^{1/2}} \right] +$$

$$ik\int_{u_1}^{\infty}\left(\frac{u}{(1+u^2)^{1/2}}\right)exp\,(-ik_1u)\,du \tag{269}$$

The term $exp\,(-ik_1\infty)$ is somewhat meaningless. We fold this back in the integral

$$I_1 = -exp\,(-ik_1u_1)\left(\frac{u_1}{(1+u_1^2)^{1/2}}\right) + e^{-ik_1u_1} +$$

$$ik\int_{u_1}^{\infty}\left(-1+\frac{u}{(1+u^2)^{1/2}}\right)exp\,(-ik_1u)\,du \tag{270}$$

This can be abbreviated as follows.

$$I_1 = exp\,(-ik_1u_1)\left[1-\left(\frac{u_1}{(1+u_1^2)^{1/2}}\right)+(-ik_1J_1)\right] \tag{271}$$

where

$$J_1 = exp\,(ik_1u_1)\int_{u_1}^{\infty}\left[1-\frac{u}{(1+u^2)^{1/2}}\right]exp\,(-ik_1u)\,du \tag{272}$$

Laschka[1] provides the following accurate approximation for $u \geq 0$

$$\left[1 - \left(\frac{u}{(1+u^2)^{1/2}}\right)\right] \approx \sum_{n=1}^{11} a_n e^{-ncu} \tag{273}$$

where $c=0.372$ and the a_n are given in the following table

n	a_n
1	+0.24186198
2	-2.7918027
3	+24.991079
4	-111.59196
5	+271.43549
6	-305.75288
7	-41.183630
8	+545.98537
9	-644.78155
10	+328.72755
11	-64.279511

Substitute equation (273) into equation (272) to obtain an approximate expression for J_1

$$J_1 \approx \sum_{n=1}^{11} \left[\frac{a_n e^{-ncu_1}}{n^2c^2 + k_1^2}\right](nc - ik_1) \tag{274}$$

We use equation (274) in equation (271) to obtain a formula for I_1. This formula is valid only for $u_1 > 0$. We see that the integrand of I_1 in equation (264) is sym-

1. See Laschka as pointed out by Giesing et. al. on page 55 of Part I, Vol I.

metric. For $u_1 < 0$ we can take advantage of the symmetry of the integrand and still use the algorithm of equation (274). We evaluate the real and imaginary components separately. For $u_1 < 0$:

$$I_1(u_1, k_1) = 2Re[I_1(0, k_1)] - Re[I_1(-u_1, k_1)] + iIm[I_1(-u_1, k_1)] \quad (275)$$

SECTION XIV

The Doublet Lattice Method

There is no precise definition of the doublet lattice method and the associated formulae. Basically, this Section restricts the approach[1] of Giesing et. al. to planar wings. In addition, we deviate in the treatment of the sweep angle. This deviation will be identified.

For a wing in the $z = 0$ plane, we have the following integral formula from equation (224).

$$\bar{w}(x, y, 0) = \frac{-1}{4\pi\rho U}\iint_S \Delta p K((x-\xi), (y-\eta), 0)\, d\xi d\eta \tag{276}$$

with the following supplementary formulae from equations (266), (262), (264), (231) and (238). Here, we have substituted ε for z to emphasize the limitation process.

$$K(x_o, y_o, 0) = \lim_{\varepsilon \to 0}\left(\frac{K_1}{y_o^2 + \varepsilon^2}\right) exp\left[\frac{-i\omega x_o}{U}\right] \tag{277}$$

$$K_1 = -I_1 - \left[\frac{M|y_o|}{(x_o^2 + \beta^2 y_o^2)^{1/2}}\right]\left[\frac{exp(-ik_1 u_1)}{(1+u_1^2)^{1/2}}\right] \tag{278}$$

1. See Giesing, Kalman and Rodden.

$$I_1 = \int_{u_1}^{\infty} \left[\frac{exp\,(-ik_1 u)}{(1+u^2)^{3/2}} \right] du \tag{279}$$

$$k_1 = \frac{\omega |y_o|}{U} \tag{280}$$

$$u_1 = \frac{M\,(x_o^2 + \beta^2 y_0^2)^{1/2} - x_o}{|y_o|\,\beta^2} \tag{281}$$

The Kernel function in equation (277) is 0^{-2} singular as y_o^2 goes to zero. This requires special consideration as one integrates equation (276). This singularity occurs when the y and η coordinates are the same. Furthermore K_1 is singular when x_o and y_o are both zero. This occurs when (x, y) is coincident with (ξ, η). Finally, as u_1 ranges from $-\infty$ to $+\infty$, we must give special consideration to equation (279). The following question remains. Can we evaluate the integral equation (276) with these mathematical difficulties. All these difficulties can be overcome analytically if one uses approximating functions. In this analytic approach, one appeals to principle values[1]. The principle values procedure requires the limit as $\varepsilon \rightarrow 0$ be taken as the absolutely final step. If one is ever in doubt about this procedure, it may prove reassuring to evaluate the integrand for several cases and plot the value as the integrand approaches the limiting singularity.

The doublet lattice method is an empirical device which simplifies the integration of this singularity. But primarily, the advantage gained by the doublet lattice method is the relative simplicity in the resulting computer program, especially for complex configurations. There are other methods which are not as simple to implement. On the other hand, there is a simpler method called the doublet point method[2]. With the doublet lattice method, the continuous pressure doublet sheet

1. See Mangler

in equation (276) is replaced by a set of pressure doublet lines with finite length. In figure 3 we picture the pattern of nine doublet lines for a rectangular wing. Each line is contained in its own box. The simplicity is that all boxes are treated identically, regardless of its proximity to the wing boundary (i.e. leading edge, trailing edge or wing tip). Other methods, based on a continuous doublet distribution require special square root singularities in the pressure distribution near the wing boundaries.

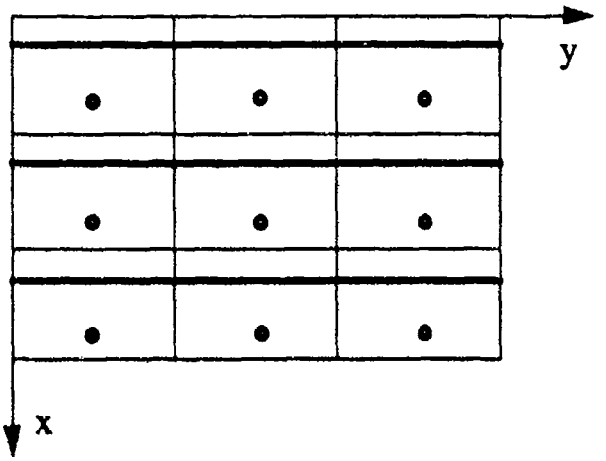

Figure 3. A Rectangular Lattice

The doublet line is placed at the quarter chord of each box. (To call this a "doublet lattice" is really a misnomer. If one views the doublet line segments alone, no lattice is formed. The name "doublet lattice" arises from the correctly named "vortex lattice" methods[1] applicable to unsteady incompressible or steady compressible flow over planar wings.) The upwash $w(x, y, 0)$ is evaluated at the 3/4 chord midspan of each box. The empirical nature of the doublet lattice method

2. See Ueda and Dowell
1. See James

arises in the choice of the 3/4 chord and 1/4 chord. By no means is there any mathematical proof that this is the correct location. As a matter of fact, for non rectangular wings with swept and tapered boxes, the programmer is left to his own devices[1] to invent meaningful 1/4 chords and 3/4 chords. However, in defense of the doublet lattice method, this level of empiricism is probably consistent with the level of approximation employed in formulating the linear aerodynamic potential equation (42). One must be on guard and realize that the flow field generated by this lattice of doublets will not be smooth, especially near the wing surface. What is important is that the upwash at the 3/4 chord is approximately the same whether one has a constant strength doublet line at the 1/4 chord or has a continuous doublet sheet with the correct strength.

Rather than dwell on rectangular wings, we assume we can invent a meaningful location for the doublet line and the upwash point for general trapezoidal boxes. A discretized swept and tapered wing is pictured here in Figure 4.

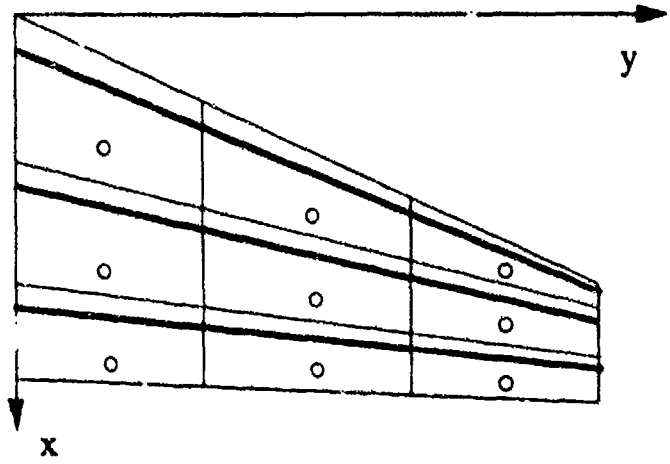

Figure 4. A Swept and Tapered Lattice

1. For instance, Giesing et. al. used a hybrid approach. In their calculations. They start with a swept doublet line. However, the resulting integral formula is carried out over an unswept doublet line. This points out the empiricism of the doublet lattice method.

So the area integral of equation (276) is truncated to a line integral along the 1/4 chord of each box. This line is depicted in Figure 5. Furthermore, we assume Δp is spatially constant for each box. For a rectangular box, the box chord is denoted as $\Delta\xi$. Equation (276) becomes

$$\bar{w}(x, y, 0) = \frac{-\Delta p\,(\Delta\xi)}{4\pi\rho U} \lim_{\varepsilon \to 0} \int_{l_1}^{l_2} K((x-\xi), (y-\eta), \varepsilon)\, dl \qquad (282)$$

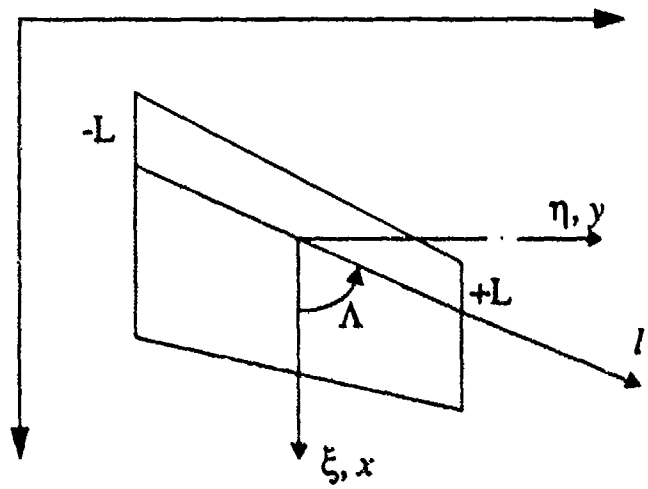

Figure 5. Local Swept Coordinates

Substituting for K from equation (277) gives

$$\bar{w}(x, y, 0) =$$

$$\left[\frac{-\Delta p \Delta\xi}{4\pi\rho U}\right] \lim_{\varepsilon \to 0} \int_{l_1}^{l_2} \frac{K_1((x-\xi), (y-\eta))\, exp\left[\frac{-i\omega(x-\xi)}{U}\right]}{(y-\eta)^2 + \varepsilon^2} dl \qquad (283)$$

Let $l_1 = -L$ and $l_2 = L$. We abbreviate equation (283) as follows

$$\bar{w}(x, y, o) = \left[\frac{-\Delta p \Delta \xi}{4\pi\rho U}\right] \lim_{\varepsilon \to 0} \int_{-L}^{L} \left[\frac{\bar{K}((x-\xi), (y-\eta))}{(y-\eta)^2 + \varepsilon^2}\right] dl \qquad (284)$$

where

$$\bar{K}(x_o, y_o, 0) = K_1(x_o, y_o) \exp\left[\frac{-i\omega x_o}{U}\right] \qquad (285)$$

Now, it is clear $\bar{K}$ is a complex function. It turns out that $\bar{K}$ can be approximated with a complex parabolic function of l.

$$\bar{K}(x_o, y_o) = A_o + A_1 l + A_2 l^2 \qquad (286)$$

where A_0, A_1 and A_2 are complex coefficients. We identify the coordinate (x_L, y_L) to represent (x_o, y_o) at $l = -L$. At the opposite end, we identify the coordinate (x_R, y_R) to represent (x_o, y_o) at $l = L$. Finally, at the midpoint, we use (x_C, y_C) to represent (x_o, y_o) at $l = 0$. Equation (286) can be formulated as

$$\bar{K}(x_o, y_o) = \left[\frac{l(l-L)}{2L^2}\right] \bar{K}(x_L, y_L) + \left[\frac{L^2 - l^2}{L^2}\right] \bar{K}(x_C, y_C) +$$

$$\left[\frac{l(L+l)}{2L^2}\right] \bar{K}(x_R, y_R) \qquad (287)$$

which is easily verified at $l = -L$, $l = 0$ and $l = L$. We regroup equation (287) in terms of common powers of l. Thus, we can identify the coefficients in equation (286) as follows

$$A_0 = \bar{K}(x_C, y_C) \tag{288}$$

$$A_1 = \frac{\bar{K}(x_R, y_R) - \bar{K}(x_L, y_L)}{2L} \tag{289}$$

$$A_2 = \frac{\bar{K}(x_L, y_L) - 2\bar{K}(x_C, y_C) + \bar{K}(x_R, y_R)}{2L^2} \tag{290}$$

Now we substitute equation (286) into equation (284). Furthermore, we note $\eta = l\sin\Lambda$.

$$\bar{w}(x, y, 0) = \left[\frac{-\Delta p \Delta\xi}{4\pi\rho U}\right] \lim_{\varepsilon \to 0} \int_{-L}^{L} \left[\frac{A_o + A_1 l + A_2 l^2}{(y - l\sin\Lambda)^2 + \varepsilon^2}\right] dl \tag{291}$$

We abbreviate equation (291) in the following fashion.

$$\bar{w}(x, y, 0) = \left[\frac{-\Delta p \Delta\xi}{4\pi\rho U}\right] [B_0 + B_1 + B_2] \tag{292}$$

The definitions of B_0, B_1 and B_2 are given as equations (293), (297) and (299).

$$B_0 = \lim_{\varepsilon \to 0} \int_{-L}^{L} \left[\frac{A_o}{(\sin\Lambda)^2 l^2 - (2y\sin\Lambda) l + (y^2 + \varepsilon^2)}\right] dl \tag{293}$$

We integrate equation (293) to obtain the following inverse tangent function

$$B_0 = \lim_{\varepsilon \to 0} \left[\frac{A_o}{\varepsilon\sin\Lambda}\right] \operatorname{atan}\left[\frac{l\sin\Lambda - y}{\varepsilon}\right]_{-L}^{L} \tag{294}$$

We use the standard tangent identity for the difference of two angles to simplify equation (294). We obtain

$$B_0 = \lim_{\varepsilon \to 0}\left[\frac{A_o}{\varepsilon \sin\Lambda}\right]\operatorname{atan}\left[\frac{2\varepsilon L \sin\Lambda}{\varepsilon^2 + y^2 - L^2(\sin\Lambda)^2}\right] \tag{295}$$

Now we take the limit.

$$B_0 = \left[\frac{2LA_o}{y^2 - L^2(\sin\Lambda)^2}\right] \tag{296}$$

We follow a similar procedure for to obtain algebraic expression for B_1 and B_2. The definition of B_1 is

$$B_1 = \lim_{\varepsilon \to 0}\int_{-L}^{L}\left[\frac{A_o l}{(\sin\Lambda)^2 l^2 - (2y\sin\Lambda)l + (y^2 + \varepsilon^2)}\right]dl \tag{297}$$

After integrating, we take the limit and obtain the following formula.

$$B_1 = \left[\frac{A_1}{2(\sin\Lambda)^2}\right]\log\left[\frac{(\sin\Lambda)^2 L^2 - (2y\sin\Lambda)L + (y^2)}{(\sin\Lambda)^2 L^2 + (2y\sin\Lambda)L + (y^2)}\right] +$$

$$\frac{yA_1}{\sin\Lambda}\left[\frac{2L}{y^2 - L^2(\sin\Lambda)^2}\right] \tag{298}$$

The definition of B_2 is

$$B_2 = \lim_{\varepsilon \to 0}\int_{-L}^{L}\left[\frac{A_2 l^2}{(\sin\Lambda)^2 l^2 - (2y\sin\Lambda)l + (y^2 + \varepsilon^2)}\right]dl \tag{299}$$

Again, after integrating, we take the limit and obtain the following formula.

$$B_2 = \left[\frac{2LA_2}{(\sin\Lambda)^2}\right] + \left[\frac{2yA_2}{\sin\Lambda}\right]\log\left[\frac{(\sin\Lambda)^2L^2 - (2y\sin\Lambda)L + (y^2)}{(\sin\Lambda)^2L^2 + (2y\sin\Lambda)L + (y^2)}\right] +$$

$$\left[\frac{y^2A_2}{(\sin\Lambda)^2}\right]\left[\frac{2L}{y^2 - L^2(\sin\Lambda)^2}\right] \qquad (300)$$

The values for B_0, B_1 and B_2 are substituted in equation (292) to obtain the relationship between $\Delta\bar{p}$ of one element and the $\bar{w}$ generated at the control point of another element. Again, $\Delta\xi$ is the average chord of an element box.

It is important for the doublet lattice user to understand the approximations incurred in discretizing a doublet sheet into trapezoidal boxes. It should be immediately obvious that, since we have assumed the pressure to be constant within each box, a sufficient number of boxes is required to capture the steady state (zero frequency) pressure function accurately. It is not obvious that we need to increase the number of boxes as we increase the frequency of oscillation. For instance, the pressure field over a rigid wing, plunging at high frequency, is not trivial and requires a significant number of boxes to resolve the standing (pressure) waves. The required box density depends on a combination of wing deformation and the frequency of motion. The box density should be increased as the deformation becomes more spatially wavy and as the temporal frequency of motion increases. The doublet lattice user must perform convergence studies to determine the appropriate box density for their application.

SECTION XV

The Example Program

The purpose of this section is to introduce a clear and simple version of a doublet lattice computer code. The mathematics of this text are tedious. Unless one is somehow inspired, these mathematics seem to exceed the bounds of reasonableness. It is possible that one may overcome this hurdle by browsing through a well annotated version of a doublet lattice code. Other existing computer codes for full aircraft configurations are difficult to follow because of the programming details. Clearly, this is not a criticism of the usefullness of these codes. Afterall, the aerospace community has depended on them for over twenty years now. They function well for a wide variety of configurations.

At the beginning of this effort to provide a tutorial on the doublet lattice method, this author had a hope that he could start with an existing code, simplify it to the planar case and then add comments. It turned out that it was more effort than was warrented. This author decided that the algorithm was so conceptually simple that he would develop his own code. Afterall, the whole raison d'etre for the doublet lattice method is that it is relatively easy to encode on a computer.

The choice of computer language was not easily made. While there is a tremendous sentiment for engineers to use FORTRAN, it is nowhere near the most overall popular computer language. The C language is very popular, especially on personal computers (PC), and it is adequate for encoding the doublet lattice method. The most important feature of the C language, as far as this tutorial is concerned, is the easy to read format. Comments can be placed just about anywhere. It is much easier to point out the relationship between the lines of the computer code to the material of this text. In addition, the C language is very

popular within the computer graphics community. With the doublet lattice method encoded in C, it is far easier for the PC programmer to connect it with a graphics library. This author believes that some day, a visually enhanced version of a doublet lattice code on a PC computer will be used to effectively motivate students toward the study of unsteady aerodynamics. The main advantage of the FORTRAN language is the COMPLEX data type. There is no equivalent data type in C. This author chose to accept this shortcoming and use the C language.

The source code for this doublet lattice code for a simple trapezoidal wing is given in Appendix A. The comments within the source code are sufficient for one to relate to the equations of this report. The example input and example output are given in Appendicies B and C. This example case is for a simple plunging rectangular wing of aspect ratio two.

As mentioned earlier, there is no strict mathematical basis for the doublet lattice method. It seems to work for rectangular wings. For swept and tapered wings, we define what we mean by the 1/4 chord and the 3/4 chord. Geising et. al. seem to use a hybrid approach. The integration process treats the line doublet as though it was not swept. However, the Kernel function is evaluated at three points along the swept doublet line. The approach taken in Appendix A is to integrate along a swept doublet line. The effect of changing the sweep of the doublet line may be an interesting topic for study.

In a sense, the doublet lattice method is empirical. Giesing et. al. point out that for steady state analysis, the integral formulae can be integrated "exactly". In order to achieve increased mathematical accuracy (this does not guarantee that correlation with test data will improve) Giesing et. al. chose to subtract out the steady state component computed by the doublet lattice method and replace this component with "exact" computations. While this could be done in Appendix A, it wasn't. The point of Appendix A was to explain the implementation of the doublet lattice method without added complication. The reader should be able to see how to implement a correction to the steady state component. However, this

author is not ready to say that anything is really gained for the effort. This is another suggested topic for study.

All the equations presented in this report made no mention of the units of measure. It turns out that there may be a benefit in non-dimensionalizing these equations. The non-dimensional solution depends on the Mach number, the reduced (non-dimensional) frequency, the shape of the wing planform and the non-dimensional deformation. The solution can then be scaled to meet a variety of different conditions such as vehicle velocity and air density. Therefore, at the expense of possibly complicating the interpretation of Appendix A, all the variables are assumed to be non-dimensional. Non-dimensional time $\bar{t}$ is scaled by

$$\bar{t} = \left[\frac{U}{b}\right] t \tag{301}$$

and non-dimensional length $\bar{x}$ is scaled by the characteristic length b

$$\bar{x} = \frac{x}{b} \tag{302}$$

The reader should have no trouble in fomulating non-dimensional upwash w. One merely divides the dimensional upwash by the freestream velocity U.

The example case is for a simple rectangular wing with aspect ratio of two. Only half of the wing is modelled. The wing is symmetric about the x axis in all respects. The wing is plunged with a reduced frequency of one. There can be no correction for steady state because the zero frequency load is zero. The solution agrees with data computed with the method of Giesing et. al.

The point of this report is not to provide a detailed explanation of the implementation. The point is to compile all the mathematics which lead to the doublet lattice method in one single document. This has been done. The utility of the code in Appendix A is not assured. The author of this report decided to include this code with the hope that its mere inclusion would help illuminate the doublet lattice method. The reader should feel free to use the example source code provided here as a starting point for developing their own utility. However, before

doing so, one should give serious consideration to using the code of Geising et. al. As indicated, their code is very versitile and is well proven. It has been the mainstay of aeroelastic analysis and design for two decades.

SECTION XVI

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APPENDIX A

The Doublet Lattice Program Source Code

definitions

```
/* the following parameters may be adjusted */
#define MAX_DIV_X 20/* maximum number of chordwise boxes */
#define MAX_DIV_Y 20/* maximum number of spanwise boxes */
#define MAX_POLY 20/* maximum number of polynomial coeff */
#define PAUSE_ON_OUTPUT 2 /* time to pause for reading output */
/* Remember to recompile after adjusting the above */

/* do not adjust the following parameters: */
#define MAXBOX (MAX_DIV_X * MAX_DIV_Y)
#define MAXDIM (MAXBOX * MAXBOX)
#define ABS(x) (((x)<0) ? -(x) : (x))
#define PI (3.141592653589793)
#define EPS (1.0e-6)
#define BIGP (1.0e+20)
#define BIGM (-1.0e+20)
/* end of define */
```

structures

```
struct element
{
 float xi,yi;/* coord of inboard 1/4 chord */
 float xm,ym;/* coord of midspan of 1/4 chord */
 float xo,yo;/* coord of outboard 1/4 chord */
 float xc,yc;/* coord of control point at 3/4 chord */
 float chord,area;/* box chord and area */
 float xcent,ycent;/* x and y coord of centroid */
};
struct trapezoid
{
 int symm;
 int   num_box_x, num_box_y, total_boxes;
```

```
 float xible,yible;
 float xibte,yibte;
 float xobte,yobte;
 float xoble,yoble;
 float mean_chord,area;
};
struct polynomial
{
 float a;
 int px,py;
};
```

MAIN: dl()

```
/*****************************************************************
* This is a doublet lattice code for a single trapezoidal wing. *
* You can assume symmetry or anti-symmetry about the x axis.    *
* This code was written by Max Blair of USAF Wright Labortory.  *
* Neither Dr Blair or the USAF assume legal responsibility for  *
* potential errors which exist in this computer code. The user  *
* is encouraged to validate the code for his or her design cases*
* of interest. Send comments to:                                *
*                                                               *
*     Dr Max Blair                                              *
*     WL/FIBRC                                                  *
*     WPAFB, OH 45433-6553                                      *
*                                                               *
* This code was written primarily for educational purposes. For *
* complete aircraft configurations, the user is encouraged to   *
* use the doublet lattice codes H7WC and N5KA.                  *
*                                                               *
* The input is placed in files dl.INPUT and bc.INPUT            *
*                                                               *
* dl.INPUT:                                                     *
* BLAIRCRAFT 2100 ATTACK FIGHTER(title line)                    *
* 5.0          characteristic length (b)                        *
* 0.5          Mach                                             *
* 1.0          reduced frequency wb/U                           *
* s            s: symmetric a: anti-symmetric  n: no symmetry   *
*  0.0  0.0    x and y coord of inboard leading edge            *
* 10.0  0.0    x and y coord of inboard trailing edge           *
* 10.0 10.0    x and y coord of inboard leading edge            *
*  0.0 10.0    x and y coord of outboard leading edge           *
* 10           number of chordwise cuts (discretized x)         *
* 10           number of spanwise cuts (discretized y)          *
```

```
*                                                                     *
* bc.INPUT:                                                          *
* flag   constant   x power   y power                                *
* 1      -1.0       0         0                                      *
* 0      -1.0       1         0                                      *
* 0      -1.0       0         1                                      *
* 0      -1.0       2         0                                      *
* 0      -1.0       1         1                                      *
* 0      -1.0       0         2                                      *
* end of data                                                        *
*                                                                     *
* interpretation of bc.INPUT:                                        *
* w(x,y) = a00 + a10*x + a01*y + a20*x^2 + a11*x*y + a02*y^2         *
* where w has been non-dimensionalized by the velocity, U.           *
* Only lines with "1" in the first column is considered data.        *
* Replace the "1" with a "0" to ignore any data.                     *
* A line which begins with an "e" will terminate the input.          *
* There must be at least one line which begins with an "e".          *
*                                                                     *
* NON-DIMENSIONAL INPUT:                                             *
* If wing coordinates are already normalized with respect to         *
* a characteristic length, then input b=1.                           *
*                                                                     *
* DIMENSIONAL INPUT:                                                 *
* If wing coordinates are input in other units (such as inches)      *
* then input any value for b such as the mean aerodynamic chord      *
* in consistent units (inches). The upwash is input in               *
* non-dimensional form, normalized with respect to the free          *
* stream velocity. Non-dimensional pressure coefficient will         *
* be printed out.                                                    *
*                                                                     *
* Output:                                                            *
* complex modulus of the pressure coefficient (Cp) at each box       *
* Cp = pressure/(density*velocity squared)                           *
**********************************************************************/

#include "dl.define"
#include <math.h>
#include <stdio.h>
#include "dl.structure"

main()
{
FILE *fopen();
FILE *aicdat, *odat;/* pointers to I/O files */
int discretize(),read_input(),Kbar(),bc();
```

```
unsigned seconds;
int i,symm,ic;
int ierr; /* error code */
int rb,sb; /* receiving box and sending box indicies*/
int vector_index; /* how the [D] matrix is placed in a vector */
float x0,y0; /* distance from receiving to sending coordinates */
float M, k; /* Mach and reduced frequency */
float b, b2; /* characteristic length (also b^2) */
float u1,beta2; /* u1 is eqtn 281 */
float KbarrL,KbariL; /* real and imag Kbar for left point */
float KbarrC,KbariC; /* real and imag Kbar for center point */
float KbarrR,KbariR; /* real and imag Kbar for right point */
float L,L2,dLx,dLy;  /* related to length of the doublet line */
float Yt,Yt2;        /* y relative to the sending midpoint */
float s1,s12,s13;    /* sine of the doublet line angle (fig 5)*/
float factor,fact0,fact1,fact2,fact3; /* working space */
float A0r,A0i,A1r,A1i,A2r,A2i; /* equation (xxx) in the text */
float B0,B1,B2; /* equation (zzz) in the text */
float wr[MAXBOX],wi[MAXBOX]; /* real and imag upwash at boxes */
float Dr[MAXDIM],Di[MAXDIM]; /* The real and imag AIC matrix */
float liftr,lifti; /* The complex lift in cartesian form */
float liftm,liftp; /* The complex lift in polar form */
struct element box[MAXBOX]; /* box geometric data */
struct trapezoid wing; /* wing geometric data */

seconds = PAUSE_ON_OUTPUT; /* seconds the program will pause */

if((odat=fopen("dl.TRASH","w"))== NULL)
 {
  printf("\ncannot open file dl.TRASH for output\n");
  exit(0);
 }
printf("\n\n Auxillary data placed in file [dl.TRASH]\n");

if((aicdat=fopen("dl.AIC","w"))== NULL)
 {
  printf("\ncannot open file dl.AIC for output\n");
  exit(0);
 }
printf("\n AIC placed in file [dl.AIC]\n");
printf("\n MAXBOX: %d MAXDIM: %d\n",MAXBOX,MAXDIM);

/* BEGIN INPUT */

printf("\nBegin input\n");
ierr = read_input(odat,&M,&k,&b,&wing);
```

```
printf("Input complete\n");
ierr = quadrilateral(odat,wing.xible,wing.yible,
                          wing.xibte,wing.yibte,
                          wing.xobte,wing.yobte,
                          wing.xoble,wing.yoble,
                          &factor,&x0,&y0);
wing.area = factor;
printf("\nWing area used to non-dimensionalize lift is:%12.4e\n",
          wing.area);
printf("The wing centroid is at x: %f and y: %f\n",x0,y0);
sleep(seconds);

/* INPUT IS NOW COMPLETE , DISCRETIZE THE WING INTO BOXES */

printf("Begin discretizing the wing.\n");
ierr = discretize(odat,wing,box);
printf("Discretization is now complete.\n");

/* non-dimensional variables are computed */

beta2  = 1-M*M;
b2 = b*b;
printf(" reduced freq (k): %12.4e \n",k);
printf(" beta^2 (1-M^2): %12.4e \n",beta2);

/* Zero out the [D] matrix of AIC coefficients */

for(rb=0;rb<wing.total_boxes;++rb)
 {
  for(sb=0;sb<wing.total_boxes;++sb)
   {
    vector_index = rb*wing.total_boxes+sb;
    Dr[vector_index] = 0.0;
    Di[vector_index] = 0.0;
   }
 }

ic = 0;
printf("\n");

/* PROCEED TO FORMULATE THE COMPLEX AIC MATRIX D[I,J] */

for(symm=0;symm<=ABS(wing.symm);++symm) /* consider symmetry */
{
for(rb=0;rb<wing.total_boxes;++rb) /* receiving box index */
{
```

```
for(sb=0;sb<wing.total_boxes;++sb) /* sending box index */
  {
   /* Kbar is defined as equation 285 */
   /* compute Kbar for left terminus of doublet segment */
   x0 = box[rb].xc - box[sb].xi;
   if(symm==0)y0 = box[rb].yc - box[sb].yi;
   else y0 = box[rb].yc + box[sb].yo;
   fprintf(odat,
     "\nLEFT symm: %2d rb: %2d sb: %2d x0: %12.4e y0: %12.4e\n",
     symm,rb,sb,x0,y0);
   ierr = Kbar(odat,M,k,x0,y0,&KbarrL,&KbariL);

   /* compute Kbar for midpoint of doublet segment */
   x0 = box[rb].xc - box[sb].xm;
   if(symm==0)y0 = box[rb].yc - box[sb].ym;
   else y0 = box[rb].yc + box[sb].ym;
   fprintf(odat,
    "\nCENTER symm: %2d rb: %2d sb: %2d x0: %12.4e y0:%12.4e\n",
     symm,rb,sb,x0,y0);
   ierr = Kbar(odat,M,k,x0,y0,&KbarrC,&KbariC);

   /* compute Kbar for right terminus of doublet segment */
   x0 = box[rb].xc - box[sb].xo;
   if(symm==0)y0 = box[rb].yc - box[sb].yo;
   else y0 = box[rb].yc + box[sb].yi;
   fprintf(odat,
     "\nRIGHT symm: %2d rb: %2d sb: %2d x0: %12.4e y0:%12.4e\n",
     symm,rb,sb,x0,y0);
   ierr = Kbar(odat,M,k,x0,y0,&KbarrR,&KbariR);

   dLx = box[sb].xo-box[sb].xi; dLy = box[sb].yo-box[sb].yi;
   L = sqrt((double)(dLx*dLx+dLy*dLy))/2.0;
   L2 = L*L;
   /* set the sweep angle here for the doublet line segment */
   if(symm==0) sl = (box[sb].yo-box[sb].ym)/L;
   else sl = (box[sb].ym-box[sb].yi)/L; /* left half of wing */
   sl2 = sl*sl;
   sl3 = sl*sl2;
   if(symm==0) Yt = box[rb].yc - box[sb].ym; /* local y axis */
   else Yt = box[rb].yc + box[sb].ym; /* left half of the wing */
   Yt2 = Yt*Yt;

   /* The real and imaginary components of the A coeffiecients
   of equations 288-290: */
   A0r = KbarrC;
   A0i = KbariC;
```

```
    A1r = (KbarrR-KbarrL)/(2*L);
    A1i = (KbariR-KbariL)/(2*L);
    A2r = ( KbarrL - 2.0*KbarrC + KbarrR )/(2.0*L*L);
    A2i = ( KbariL - 2.0*KbariC + KbariR )/(2.0*L*L);

    fact0 = (sl2*L2-2.0*Yt*sl*L+Yt2)/(sl2*L2+2.0*Yt*sl*L+Yt2);
    fact0 = log((double)(fact0)); /* nat log */
    B0 = (2.0*L)/(Yt2-L2*sl2); /* equation 296) */
    fact1 = (0.5/sl2)*fact0;
    fact2 = (Yt/sl)*B0;
    B1 = (fact1+fact2); /* equation 298 */
    fact1 = 2.0*L/sl2;
    fact2 = (Yt/sl3)*fact0;
    fact3 = (Yt2/sl2)*B0;
    B2 = (fact1+fact2+fact3); /* equation 300 */

    factor = (-box[sb].chord/(8.0*PI));
/*
 Here, we will compress square matricies Dr and Di into vectors.
 The matricies Dr and Di are the real and imaginary parts of [D].
 {w} = [D]{Cp}
 {w} is a vector of non-dim upwash w/U at the control points.
 {Cp} is the vector of non-dimensional pressure dP/(rho*U^2).
 [D] is compressed into {Dr} and {Di} in row packets.
*/
    vector_index = rb*wing.total_boxes+sb;
    if(symm==0) /* for any symmetric case */
     {
      Dr[vector_index] = Dr[vector_index] +
                         factor*(B0*A0r+B1*A1r+B2*A2r);
      Di[vector_index] = Di[vector_index] +
                         factor*(B0*A0i+B1*A1i+B2*A2i);
     }
    else if(symm==1&&wing.symm>0) /* left half wing is symm */
     {
      Dr[vector_index] = Dr[vector_index] +
                         factor*(B0*A0r+B1*A1r+B2*A2r);
      Di[vector_index] = Di[vector_index] +
                         factor*(B0*A0i+B1*A1i+B2*A2i);
     }
    else if(symm==1&&wing.symm<0) /* left wing is anti-symm */
     {
      Dr[vector_index] = Dr[vector_index] -
                         factor*(B0*A0r+B1*A1r+B2*A2r);
      Di[vector_index] = Di[vector_index] -
                         factor*(B0*A0i+B1*A1i+B2*A2i);
```

```
      }
     else
      {
       printf("\n CONFUSED about the wing symmetry... \nexit\n");
       exit(0);
      }

    } /* end of loop on sb */

   if(++ic>10)
    {
     printf("\n");
     ic=0;
    }
   printf(" %d",rb);
  } /* end of loop on rb */
  } /* end of loop on symm */
printf("\n\n");

/* Print out the [D] matrix of AIC coefficients */
for(rb=0;rb<wing.total_boxes;++rb)
 {
  for(sb=0;sb<wing.total_boxes;++sb)
   {
    vector_index = rb*wing.total_boxes+sb;
    fprintf(aicdat,"row: %5d col: %5d ind: %10d %15.7e %15.7e\n",
          rb,sb,vector_index, Dr[vector_index],Di[vector_index]);
   }
 }

/* Input the {w} boundary condition, return # of monomials */
if( (ierr = bc(odat,k,wr,wi,box,wing)) == 0 )
 {
  printf("\nNo upwash specified and no pressure computed\n");
  fclose(odat);
  fclose(aicdat);
  exit(0);
 }
else
 {
  printf("\n Upwash specified and pressure will be computed\n");
 }

for(i=0;i<wing.total_boxes;++i)
 {
  printf("%5d Real[w]: %12.4e Imag[w]: %12.4e \n",
```

```
            i,wr[i],wi[i]);
    fprintf(odat,"%5d Real[w]: %12.4e Imag[w]: %12.4e \n",
            i,wr[i],wi[i]);
   }

  /* solve the complex problem {w} = [D]{p} */
  printf("\nSolve the complex problem {w} = [D]{p}\n");
  ierr=complex_solve(Dr,Di,wr,wi,wing.total_boxes);
  if(ierr!=0)
   {
    printf("\nerror number %d in complex_solve\n",ierr);
    exit(0);
   }
  /* note: the [D] matrix is now the complex identity matrix */

  /* Print out the pressure coefficients and sum the lift*/

  printf("\n\nCp PRESSURE COEFFICIENTS (P=0.5*rho*U^2*Cp)");
  printf ( "\nbox # (real Cp) (imag Cp) (box area)\n");
  fprintf(odat,"\n\nCp PRESSURE COEFFICIENTS (P=0.5*rho*U^2*Cp)");
  fprintf(odat,"\nbox # (real Cp) (imag Cp) (box area)\n");
  liftr = 0.0; lifti = 0.0;
  for(i=0;i<wing.total_boxes;++i)
   {
    printf( "%5d %12.4e %12.4e %12.4e\n",
          i,wr[i],wi[i],b2*box[i].area);
    fprintf(odat,"%5d %12.4e %12.4e %12.4e\n",
          i,wr[i],wi[i],b2*box[i].area);
    liftr += wr[i]*box[i].area;
    lifti += wi[i]*box[i].area;
   }
  liftr /= wing.area;
  lifti /= wing.area;
  ierr = polar(liftr,lifti,&liftm,&liftp);
  liftp *= (180.0/PI);
  printf("\nThe characteristic length b: ",b);
  printf("The wing area used to non-dim lift is:%12.4e\n",
         wing.area);
  printf("Lift Coefficient = C_L*q*A\n");
  printf("THE COMPLEX WING LIFT COEFFICIENT (C_L) IS: ");
  printf("[(%11.4e) + (%11.4e)i]\n",liftr,lifti);
  printf("MAGNITUDE: (%11.4e) PHASE: (%9.4f)deg\n",liftm,liftp);


  fclose(odat);
  fclose(aicdat);
```

```
exit(0);
}
```

FUNCTION: read_input()

```
#include "dl.define"
#include <math.h>
#include <stdio.h>
#include <string.h>
#include "dl.structure"

read_input(odat,M,k,b,wing)
float *M, *k, *b;
struct trapezoid *wing;
FILE *odat;
{
FILE *fopen();
FILE *idat;
char csymm;
char line[200];
int ierr;
int n;
float x, y;

if((idat=fopen("dl.INPUT","r"))== NULL)
 {
  printf("\ncannot open file dl.INPUT for input\n");
  exit(0);
 }
printf("Input data will be read from file [dl.INPUT]\n");

/* BEGIN INPUT */

if( in_line(idat,line)==0 ) /* read title line */
 {
  fprintf(odat,"\ntext: [%s]",line);
  printf("TITLE:\ntext: [%s]\n",line);
 }
if( in_line(idat,line)==0 ) /* read characteristic length */
 {
  sscanf(line,"%f",&x);
  *b = x;
  printf("characteristic length: %f\n",*b);
  fprintf(odat,"characteristic length: %f\n",*b);
 }
if( in_line(idat,line)==0 ) /* read Mach number */
```

```
  {
   sscanf(line,"%f",&x);
   *M = x;
   printf("Mach: %f\n",*M);
   fprintf(odat,"Mach: %f\n",*M);
  }
 if( in_line(idat,line)==0 ) /* read reduced frequency */
  {
   sscanf(line,"%f",&x);
   *k = x;
   printf("reduced frequency: %f\n",*k);
   fprintf(odat,"reduced frequency: %f\n",*k);
  }
 if( in_line(idat,line)==0 ) /* read symmetry flag -1, 0 or +1 */
  {
   sscanf(line,"%c",&csymm);
   line[0] = csymm; line[1] = '\0';
   wing->symm = 0;
   if( strcmp(line,"s")==0 )
    {
     wing->symm = 1;
     printf("Assume symmetry about the x axis\n");
     fprintf(odat,"Assume symmetry about the x axis\n");
    }
   else if( strcmp(line,"a")==0 )
    {
     wing->symm = -1;
     printf("Assume anti-symmetry about the x axis\n");
     fprintf(odat,"Assume anti-symmetry about the x axis\n");
    }
     else
    {
     wing->symm = 0;
     printf("Assume no symmetry about the x axis\n");
     fprintf(odat,"Assume no symmetry about the x axis\n");
    }
  }
 if( in_line(idat,line)==0 ) /* read inbrd lead edge coord */
  {
   sscanf(line,"%f %f",&x,&y);
   wing->xible = x;
   wing->yible = y;
   printf("inboard leading edge: x %f y %f\n",
          wing->xible,wing->yible);
   fprintf(odat,"inboard leading edge: x %f y %f\n",
          wing->xible,wing->yible);
```

```
  wing->xible /= *b;
  wing->yible /= *b;
 }
if( in_line(idat,line)==0 ) /* read inbrd trail edge coord */
 {
  sscanf(line,"%f %f",&x,&y);
  wing->xibte = x;
  wing->yibte = y;
  printf("inboard trailing edge: x %f y %f\n",
         wing->xibte,wing->yibte);
  fprintf(odat,"inboard trailing edge: x %f y %f\n",
         wing->xibte,wing->yibte);
  wing->xibte /= *b;
  wing->yibte /= *b;
 }
if( in_line(idat,line)==0 ) /* read outbrd trail edge coord */
 {
  sscanf(line,"%f %f",&x,&y);
  wing->xobte = x;
  wing->yobte = y;
  printf("outboard trailing edge: x %f y %f\n",
         wing->xobte,wing->yobte);
  fprintf(odat,"outboard trailing edge: x %f y %f\n",
         wing->xobte,wing->yobte);
  wing->xobte /= *b;
  wing->yobte /= *b;
 }
if( in_line(idat,line)==0 ) /* read outbrd lead edge coord */
 {
  sscanf(line,"%f %f",&x,&y);
  wing->xoble = x;
  wing->yoble = y;
  printf("outboard leading edge: x %f y %f\n",
         wing->xoble,wing->yoble);
  fprintf(odat,"outboard leading edge: x %f y %f\n",
         wing->xoble,wing->yoble);
  wing->xoble /= *b;
  wing->yoble /= *b;
 }
if( in_line(idat,line)==0 ) /* read number of boxes in x */
 {
  sscanf(line,"%d",&n);
  wing->num_box_x = n;
  printf("number of boxes in the x direction: %d\n",
         wing->num_box_x);
  fprintf(odat,"number of boxes in the x direction: %d\n",
```

```
            wing->num_box_x);
    if(wing->num_box_x > MAX_DIV_X)
     {
      printf("\n EXCEEDED MAXIMUM DIMENSION ON DIVISIONS IN X\n");
      exit(0);
     }
   }
 if( in_line(idat,line)==0 ) /* read number of boxes in y */
  {
    sscanf(line,"%d",&n);
    wing->num_box_y = n;
    printf("number of boxes in the y direction: %d\n",
          wing->num_box_y);
    fprintf(odat,"number of boxes in the y direction: %d\n",
          wing->num_box_y);
    if(wing->num_box_y > MAX_DIV_Y)
     {
      printf("\n EXCEEDED MAXIMUM DIMENSION ON DIVISIONS IN Y\n");
      exit(0);
     }
  }

wing->total_boxes = wing->num_box_x * wing->num_box_y;
if(wing->total_boxes>MAXBOX)
 {
  printf("\n EXCEEDED MAXIMUM DIMENSION ON BOXES\n");
  exit(0);
 }

fclose(idat);

/* INPUT IS NOW COMPLETE */

return(0);
}
```

FUNCTION: discretize()

```
/*************************************************************************
*Calculate the coordinates of the discretized wing.                     *
*************************************************************************/

#include "dl.define"
#include <math.h>
#include <stdio.h>
#include "dl.structure"
```

FUNCTION: discretize()

```
int discretize(odat,wing,box)
FILE *odat;
struct element box[];
struct trapezoid wing;
{
int quadrilateral();
int ierr;
int rx,ry;  /* x index and y index */
int box_index;
float dxib,dxob; /* delta x on the inboard and outboard chords */
float dy; /* delta y is constant over the span */
float x1,x2,x3,x4;
float y1,y2,y3,y4;
float ix,iy;
float area,total_area;

total_area = 0.0;
/* delta x along the inboard and outboard: */
dxib = (wing.xibte-wing.xible)/(float)wing.num_box_x;
dxob = (wing.xobte-wing.xoble)/(float)wing.num_box_x;
/* delta y of all boxes: */
dy = (wing.yoble-wing.yible)/(float)wing.num_box_y;
fprintf(odat,"dxib: %f dxob: %f dy: %f\n",dxib,dxob,dy);

for(ry=0;ry<wing.num_box_y;++ry) /* box y index */
{
for(rx=0;rx<wing.num_box_x;++rx) /* box x index */
{
  box_index = ry*wing.num_box_x + rx;
  /* printf("rx: %3d ry: %3d receive_index: %4d\n",
          rx,ry,box_index); */
  /* inboard leading edge coordinates of receiving box: */
  ix = (float)rx;
  iy = (float)ry/(float)wing.num_box_y;
  x1 = ix*dxib + ix*(dxob-dxib)*iy; /*ible*/
  y1 = ry*dy;
  /* inboard trailing edge coordinates of receiving box: */
  ix = (float)(rx+1);
  iy = (float)ry/(float)wing.num_box_y;
  x2 = ix*dxib + ix*(dxob-dxib)*iy; /*ible*/
  y2 = ry*dy;
  /* outboard trailing edge coordinates of receiving box: */
  ix = (float)(rx+1);
  iy = (float)(ry+1)/(float)wing.num_box_y;
  x3 = ix*dxib + ix*(dxob-dxib)*iy; /*ible*/
```

```
    y3 = (ry+1)*dy;
    /* outboard leading edge coordinates of receiving box: */
    ix = (float)rx;
    iy = (float)(ry+1)/(float)wing.num_box_y;
    x4 = ix*dxib + ix*(dxob-dxib)*iy; /*ible*/
    y4 = (ry+1)*dy;
    /* coord of the receiving control pt at 3/4 chord centerspan */
    box[box_index].xc = ( x1+0.75*(x2-x1) + x4+0.75*(x3-x4) )/2.0;
    box[box_index].yc = (y2+y3)/2.0;
    /* inboard coord of the sending doublet line along 1/4 chord */
    box[box_index].xi = x1+0.25*(x2-x1);
    box[box_index].yi = y1;
    /* outboard coord of the sending doublet line along 1/4 chord */
    box[box_index].xo = x4+0.25*(x3-x4);
    box[box_index].yo = y4;
    /* midspan coord of the sending doublet line along 1/4 chord */
    box[box_index].xm = (box[box_index].xi+box[box_index].xo)/2.0;
    box[box_index].ym = (box[box_index].yi+box[box_index].yo)/2.0;
    /* average chord */
    box[box_index].chord = ((x2-x1)+(x3-x4))/2.0;
    /* box area and x and y coordinates of the box centroid */
    ierr = quadrilateral(odat,x1,y1,x2,y2,x3,y3,x4,y4,
                         &area,&ix,&iy);
    box[box_index].area = area;
    box[box_index].xcent = ix;
    box[box_index].ycent = iy;
    total_area += area;
    fprintf(odat,"\n BOX: %5d\n",box_index);
    fprintf(odat," x1: %8.4f x2: %8.4f x3: %8.4f x4: %8.4f\n",
            x1,x2,x3,x4);
    fprintf(odat," y1: %8.4f y2: %8.4f y3: %8.4f y4: %8.4f\n",
            y1,y2,y3,y4);
    fprintf(odat," 3/4 chord midspan x: %f y: %f\n",
            box[box_index].xc,box[box_index].yc);
    fprintf(odat," 1/4 chord inboard x: %f y: %f \n",
            box[box_index].xi,box[box_index].yi);
    fprintf(odat," 1/4 chord midspan x: %f y: %f \n",
            box[box_index].xm,box[box_index].ym);
    fprintf(odat," 1/4 chord outboard x: %f y: %f \n",
            box[box_index].xo,box[box_index].yo);
    fprintf(odat," average chord: %f \n",box[box_index].chord);
    fprintf(odat," area: %f \n",area);
    fprintf(odat," x centroid: %f y centroid: %f\n",ix,iy);

  } /* end of loop on rx */
  } /* end of loop on ry */
```

```
fprintf(odat,"\nTOTAL AREA OF WING: %f\n",total_area);
return(0);
}
```

FUNCTION: Kbar()

```
/*****************************************************************
* This subroutine computes K bar.                                *
* K bar is given as equation 273 in the text).                   *
* K1 is given as equation 266 in the text.                       *
*****************************************************************/

#include "dl.define"
#include <math.h>
#include <stdio.h>

int Kbar(odat,M,k,x0,y0,Kbr,Kbi)

FILE *odat;
float M;/* mach */
float k;/* reduced freq wb/U */
float x0,y0;/* non-dimensional (x-xi)/b and (y-eta)/b */
float *Kbr,*Kbi;/* return these values */
{
int ierr,I1();
float u1,k1;
float alpha,beta2;
float K1r,K1i,factor;
float exr,exi,eur,eui;
float I1r,I1i;
float U1;

fprintf(odat,"In Kbar now\n");
fprintf(odat,"M: %12.4e\n",M);
fprintf(odat,"k: %12.4e\n",k);
fprintf(odat,"x0: %12.4e y0: %12.4e\n",x0,y0);

beta2 = 1-M*M;
k1= k*ABS(y0);
fprintf(odat,"k1: %12.4e\n",k1);
if((ABS(y0))<EPS) /* if y0 = zero, we need to take care for u1 */
 {
  if(x0>0.0){u1=BIGM;}
  else {u1=BIGP;}
 }
```

```
else /* y0 is not equal to zero */
 {
  u1 = (M*sqrt((double)(x0*x0+beta2*y0*y0))-x0)/(ABS(y0)*beta2);
 }
alpha= -x0*k;
exr= cos((double)(alpha));
exi= sin((double)(alpha));
ierr = I1(odat,u1,k1,&I1r,&I1i); /* compute the I1 integral */

/* compute K1 = (K1r + i*K1i) */
if( u1>=BIGP || u1<=BIGM )
 {
  fprintf(odat,"BIG u1: %12.4e\n",u1);
  K1r = -I1r;
  K1i = -I1i;
 }
else if( u1<BIGP && u1>BIGM )
 {
  fprintf(odat,"bounded u1: %12.4e\n",u1);
  alpha = -k1*u1;
  eur = cos((double)(alpha));
  eui = sin((double)(alpha));
  factor = (-1.0)*
           ( M*ABS(y0)/sqrt((double)((x0*x0)+(beta2*y0*y0))) )
           /sqrt((double)(1.0+u1*u1));
  K1r = factor*eur - I1r;
  K1i = factor*eui - I1i;
 }
else
 {
  printf("\nConfused with u1 in function Kbar\n");
 }

/* compute Kbar = (Kbr + i*Kbi) */
ierr = cmult(K1r,K1i,exr,exi,Kbr,Kbi);
return(ierr);
}
```

FUNCTION: I1()

```
/*************************************************************
* This soubroutine computes the integral I1.                 *
* I1 is given as equation 267 in the text                    *
* I1 computes the I1 integral for any u1.                    *
*************************************************************/
```

FUNCTION: I1()

```
#include <math.h>
#include <stdio.h>
#include "dl.define"

int I1(odat,u1,k1,I1r,I1i)

FILE *odat;
float u1,k1;/* input these values */
float *I1r,*I1i;/* return these values */

{
int ierr, cmult();
float U1, Ireal;

fprintf(odat,"(IN I1) u1: %12.4e k1: %12.4e\n",u1,k1);
if(u1>=0.0)
 {
  if(u1>=BIGP) /* +BIG < u1 */
   {
    *I1r = 0.0;
    *I1i = 0.0;
   }
  else /* 0 <= u1 < +BIG */
   {
    ierr = i1(odat,u1,k1,I1r,I1i);
   }
 }
else /* u1 < 0 */
 {
  if(u1<=BIGM) /* u1 < -BIG */
   {
    fprintf(odat,"u1 < -BIG \n");
    U1 = 0.0;
    ierr = i1(odat,U1,k1,I1r,I1i);
    *I1r *= 2.0;
    *I1i *= 2.0;
   }
  else /* -BIG < u1 < 0 */
   {
    /* compute I1 for 0<u<infinity: */
    fprintf(odat," -BIG < u1 < 0\n");
    U1 = 0.0;
    ierr = i1(odat,U1,k1,I1r,I1i);
    Ireal = *I1r;
    U1 = (-1.0)*u1;
    ierr = i1(odat,U1,k1,I1r,I1i);
```

```
    *I1r = (2.0*Ireal) - (*I1r);
   }
 }
return(ierr);
}
```

FUNCTION i1()

```
/*****************************************************************
*Function to compute the I1 integral for u1 >= 0*
*****************************************************************/

int i1(odat,u1,k1,i1r,i1i)

FILE *odat;
float u1,k1;
float *i1r,*i1i;

{
int ierr, cmult(), J1();
float ir,ii,alpha,factor,er,ei;
float j1r,j1i;

fprintf(odat,"(IN i1) u1: %12.4e k1: %12.4e\n",u1,k1);
if(u1<0.0)
 {
  printf("\n u1 cannot be less than zero...u1 = %12.4e\n",u1);
  exit(0);
 }
ierr = J1(odat,u1,k1,&j1r,&j1i);
factor = 1.0 - u1/sqrt((double)(1.0+u1*u1));
ir = factor + k1*j1i;
ii = -k1*j1r;
alpha = -k1*u1;
er = cos((double)(alpha));
ei = sin((double)(alpha));
ierr = cmult(er,ei,ir,ii,i1r,i1i);

return(ierr);
}
```

FUNCTION J1()

```
/*****************************************************************
* This subroutine computes the integral J1.                      *
```

```
* The integrand of J1 is approximated as an                    *
* algebraic expression and then integrated.                     *
* The series expression for J1 is given as                      *
* equation xxx in the text.                                     *
*****************************************************************/

#include <math.h>
#include <stdio.h>

int J1(odat,u1,k1,J1r,J1i)

FILE *odat;
float u1,k1;/* input these values */
float *J1r,*J1i;/* return these values */
{
int i,ierr;
static int n = 11;
float j,zero,sjr,sji,jr,ji;
double djr,dji;
static float c = 0.372;
static float a[11] = { 0.24186198,
                       -2.7918027,
                        24.991079,
                      -111.59196,
                       271.43549,
                      -305.75288,
                       -41.183630,
                       545.98537,
                      -644.78155,
                       328.72755,
                       -64.279511 );
fprintf(odat,"(IN J1) u1: %12.4e k1: %12.4e\n",u1,k1);
ierr = 0;
zero = 0.0;
djr = (double)zero;
dji = (double)zero;

for(i=1;i<=n;++i)
 {
  j   = a[i-1]*exp((double)(-i*c*u1))/
        ((float)(i*i)*(c*c)+(k1*k1));
  sjr = j*(float)i*c;
  sji = -j*k1;
  djr = djr + (double)sjr;
  dji = dji + (double)sji;
  fprintf(odat,"a[%2d]: %12.4e\n",i,a[i-1]);
```

```
 }

*J1r = (float)djr;
*J1i = (float)dji;

return(ierr);
}
```

FUNCTION bc()

```
#include "dl.define"
#include <math.h>
#include <stdio.h>
#include <string.h>
#include "dl.structure"

int bc(odat,k,wr,wi,box,wing)
FILE *odat;
float k;
float wr[],wi[];
struct element box[];
struct trapezoid wing;
{
FILE *fopen();
FILE *idat;
char c,flag[2],line[200];
int ierr,do_flag,number;
int i,j,px,py;
float a, x, y, sumr, sumi;
float power();
struct polynomial poly[MAX_POLY];

if((idat=fopen("bc.INPUT","r"))== NULL)
 {
  printf("\ncannot open file bc.INPUT for input\n");
  exit(0);
 }
printf("Boundary data will be read from file [dl.INPUT]\n");
printf("reduced frequency (k): %12.4e\n",k);

number = 0;

/* BEGIN INPUT */

do{
  do_flag = 1;
```

```
  if( in_line(idat,line)==0 )
   {
    /* fprintf(odat,"\ntext: [%s]",line); */
    /* printf("text: [%s]\n",line); */
    sscanf(line,"%c",&c);
    flag[0]=c;
    flag[1]='\0';
    if( strcmp(flag,"e")==0 )/* end of data */
     {
      do_flag=0;
     }
    if( strcmp(flag,"1")==0 )/* a line of data is in line[] */
     {
      sscanf(line,"%c %f %d %d",&c,&a,&px,&py);
      poly[number].a = a; /* coefficient */
      poly[number].px = px; /* power of x */
      poly[number].py = py; /* power of y */
      printf(" [%c] a: %f px: %d py: %d\n",c,a,px,py);
      ++number;
      do_flag = 1;
     }
   }
}while(do_flag==1);

fclose(idat);
if(number==0)return(number); /* return zero if no data input */

/* Compute the upwash at each box 3/4 chord */

printf("\n compute the upwash at %d control points\n",
       wing.total_boxes);
for(i=0;i<wing.total_boxes;++i)
 {
  x = box[i].xc;
  y = box[i].yc;
  sumr = 0.0;
  sumi = 0.0;
  for(j=0;j<number;++j)
   {
    sumr = sumr + (poly[j].a * poly[j].px) *
                  power(x,(poly[j].px-1)) * power(y,poly[j].py);
    sumi = sumi + k * poly[j].a *
                  power(x,poly[j].px) * power(y,poly[j].py);
   }
  wr[i] = sumr;
  wi[i] = sumi;
```

```
  fprintf(odat," %5d Real[w]: %12.4e Imag[w]: %12.4e\n",
          i,wr[i],wi[i]);
}

return(number);
}
```

FUNCTION complex_solve()

```
/*
solve is a function to solve the complex linear system [a]{x}={c}
using Gaussian elimination and back substitution with pivoting
on each step. [a] is input in vector form a(k) = a(i,j) where
k=i*nc+j. nc is the utilized portion of a[nc][nc] and c[nc]
where nc <= the declared dimension. The complex solution {x}
is returned in {c}.

The real and imaginary parts of a[][] are designated as
ar[] and ai[]. Likewise for the c vector.
*/

#include <stdio.h>
#define ABS(x) (((x)<0) ? -(x) : (x))

int complex_solve(ar,ai,cr,ci,nc)
int nc;
float ar[],ai[],cr[],ci[];
{

int ierr,i,j,i2,nr,ir,jc;
int cdiv(), cmult();
float rmaxabs,tmpabs,tmpr,tmpi,dr,di,big=1.0e+20,eps=1.0e-20;
float cmag();

/* Consider nc rows: */
for(i=0;i<nc;++i)
 {

  /* find max diagonal value and switch rows */
  nr=i;
  rmaxabs=cmag(ar[i*nc+i],ai[i*nc+i]);
  for(i2=i;i2<nc;++i2)
   {
    tmpabs=rmaxabs-cmag(ar[i2*nc+i],ai[i2*nc+i]);
    if(tmpabs<0.0)
     {
```

```
      nr=i2;
      rmaxabs=cmag(ar[i2*nc+i],ai[i2*nc+i]);
     }
   } /* end i2 loop*/

  for(j=0;j<nc;++j)
   {
    tmpr=ar[nr*nc+j];
    tmpi=ai[nr*nc+j];
    ar[nr*nc+j]=ar[i*nc+j];
    ai[nr*nc+j]=ai[i*nc+j];
    ar[i*nc+j]=tmpr;
    ai[i*nc+j]=tmpi;
   }
  tmpr=cr[nr];
  tmpi=ci[nr];
  cr[nr]=cr[i];
  ci[nr]=ci[i];
  cr[i]=tmpr;
  ci[i]=tmpi;
  /* rows have been switched */

  dr=ar[i*nc+i];
  di=ai[i*nc+i];
  if(ABS(cmag(dr,di))<=eps)return(1);
  for(j=i;j<nc;++j)
   {
    ierr=cdiv(ar[i*nc+j],ai[i*nc+j],dr,di,&tmpr,&tmpi);
    ar[i*nc+j] = tmpr;
    ai[i*nc+j] = tmpi;
   }
  ierr=cdiv(cr[i],ci[i],dr,di,&tmpr,&tmpi);
  cr[i] = tmpr;
  ci[i] = tmpi;
  if( ABS(cmag(cr[i],ci[i])) >= big )return(5);

  for(ir=0;ir<nc;++ir)
   {
    if(ir==i)continue;
    dr = ar[ir*nc+i];
    di = ai[ir*nc+i];
    if( ABS(cmag(dr,di)) >= big)return(2);
    if( ABS(cmag(cr[i],ci[i])) >= big)return(4);
    for(jc=i;jc<nc;++jc)
     {
      ierr=cmult(dr,di,ar[i*nc+jc],ai[i*nc+jc],&tmpr,&tmpi);
```

```
      ar[ir*nc+jc] -= tmpr;
      ai[ir*nc+jc] -= tmpi;
     } /* end jc loop */
    ierr=cmult(dr,di,cr[i],ci[i],&tmpr,&tmpi);
    cr[ir] -= tmpr;
    ci[ir] -= tmpi;
   } /*ir loop*/

 } /* i loop*/

return(0);
}
```

FUNCTION complex_math()

```
/*****************************************************************
 This file is the source for a collections of subroutines
 written by Max Blair to perform simple complex operations.
*****************************************************************/

#include <stdio.h>
#include <math.h>
#define ABS(x) (((x)<0) ? -(x) : (x))
#define EPS (1.0e-20)
#define BIG (1.0e+20)

/* add c=a+b */
int cadd(ar,ai,br,bi,cr,ci)
float ar,ai;/* input this complex number */
float br,bi;/* input this complex number */
float *cr,*ci;/* return this complex number */
{
*cr = ar+br;
*ci = ai+bi;
return(0);
}

/* add c=a-b */
int csub(ar,ai,br,bi,cr,ci)
float ar,ai;/* input this complex number */
float br,bi;/* input this complex number */
float *cr,*ci;/* return this complex number */
{
*cr = ar-br;
*ci = ai-bi;
return(0);
```

```
}

/* multiply c=a*b */
int cmult(ar,ai,br,bi,cr,ci)
float ar,ai;/* input this complex number */
float br,bi;/* input this complex number */
float *cr,*ci;/* return this complex number */
{
*cr = (ar*br)-(ai*bi);
*ci = (ar*bi)+(ai*br);
return(0);
}

/* divide c=a/b */
int cdiv(ar,ai,br,bi,cr,ci)
float ar,ai;/* input this complex number */
float br,bi;/* input this complex number */
float *cr,*ci;/* return this complex number */
{
float d;
d = (br*br)+(bi*bi);
if(d<EPS)
 {
  printf("\n division by complex zero in cdiv\n");
  exit(0);
 }
else
 {
  *cr = ((ar*br)+(ai*bi))/d;
  *ci = ((ai*br)-(ar*bi))/d;
 }
return(0);
}

/* transforms complex number from cartesian to polar form */
int polar(ar,ai,bm,bp)
float ar,ai; /* input cartesian form of complex number a */
float *bm,*bp; /* return polar form of b, mag and phase (rad)*/
{
float ftheta();
*bm = sqrt((double)((ar*ar)+(ai*ai)));
*bp = ftheta(ar,ai);
return (0);
}

/* transforms complex number from polar to cartesian form */
```

```
int cartesian(am,ap,br,bi)
float am,ap; /* input polar form of complex number a, magnitude
and phase (rad) */
float *br,*bi; /* return cartesian form of complex number b */
{
*br = am*cos((double)ap);
*bi = am*sin((double)ap);
return (0);
}

/* return the value of theta (rad) given x and y coordinates: */
float ftheta(x,y)
float x,y;
{
float pi,xtest,theta;

pi=acos((double)(-1.0));
xtest=fabs( (double)(y*1.0e-05) );
theta=pi/2.0;
if(y<0.0)theta=(-pi)/2.0;
if(fabs((double)x)<=xtest)return(theta);
theta=atan((double)(y/x));
if(x<0.0)theta=theta+pi;
return(theta);
}

/* absolute value of complex number in cartesian form */
float cmag(ar,ai)
float ar,ai; /* input cartesian form of complex number a */
{
float mag; /* return polar form of b, magnitude and phase (rad)*/
if( ABS(ar)>BIG || ABS(ai)>BIG )
 {
  printf("potential error in cmag ar: %e ai: %e\n",ar,ai);
  if( ABS(ar)>BIG )mag = ar;
  if( ABS(ai)>BIG )mag = ai;
  return(mag);
 }
if( ABS(ar)<EPS && ABS(ai)<EPS )return(0.0);
mag = sqrt((double)((ar*ar)+(ai*ai)));
return(mag);
}
```

FUNCTION power()

```
float power(x,i)
/* carry out the operation x^i where x is real and i is integer */
int i;
float x;
{
int j;
float p;
if(i==0)return(1);
if(i==1)return(x);
p=x;
for(j=2;j<=i;j++)
{
p=p*x;
}
return(p);
}
```

FUNCTION quadrilateral()

```
/* input the coordinates in a counter-clockwise order */

#include "dl.define"
#include <math.h>
#include <stdio.h>
#include "dl.structure"

int quadrilateral(odat,x1,y1,x2,y2,x3,y3,x4,y4,area,xc,yc)
FILE *odat;
float x1,x2,x3,x4;
float y1,y2,y3,y4;
float *area, *xc, *yc;
{
float a1,a2,a3,a4;
float b1,b2,b3,b4;
float c1,c2,c3;

a1 = (-x1+x2+x3-x4)/4.0;
a2 = ( x1-x2+x3-x4)/4.0;
a3 = (-x1-x2+x3+x4)/4.0;
a4 = ( x1-x2+x3-x4)/4.0;
b1 = (-y1-y2+y3+y4)/4.0;
b2 = ( y1-y2+y3-y4)/4.0;
b3 = (-y1+y2+y3-y4)/4.0;
b4 = ( y1-y2+y3-y4)/4.0;
c1 = ( a1*b1-a3*b3);
c2 = ( a1*b2-a2*b3);
```

```
c3 = ( a2*b1-a3*b2);
*area = 4.0*c1;
if(*area<=0.0)return(1);

a1 = ( x1+x2+x3+x4)/4.0;
a2 = (-x1+x2+x3-x4)/4.0;
a3 = (-x1-x2+x3+x4)/4.0;
a4 = ( x1-x2+x3-x4)/4.0;
b1 = ( y1+y2+y3+y4)/4.0;
b2 = (-y1+y2+y3-y4)/4.0;
b3 = (-y1-y2+y3+y4)/4.0;
b4 = ( y1-y2+y3-y4)/4.0;
*xc = ( 4.0*(a1*c1) + 4.0*(a2*c2)/3.0 + 8.0*(a3*c3) )/(*area);
*yc = ( 4.0*(b1*c1) + 4.0*(b2*c2)/3.0 + 8.0*(b3*c3) )/(*area);

return(0);
}
```

FUNCTION in_line()

```
#include <stdio.h>

int in_line(idat,line)

char line[];
FILE *idat;

{
char c1;
int j;

c1=getc(idat);

if(c1==EOF)
 {
  line[0]='\0';
  return(1);
 }
else if(c1==10)
 {
  line[0]='\0';
  return(0);
 }
else
 {
  line[0]=c1;
```

```
 }

j=1;
while( (line[j]=getc(idat)) != 10 )
 {
  if(line[j]==EOF)
   {
    printf("\nEND OF FILE ENCOUNTERED IN in_line()\n");
    exit(0);
   }
  j++;
 }
line[j]='\0';
return(0);
}
```

end of doublet lattice source code

APPENDIX B

The Doublet Lattice Program Input File

dl_INPUT

```
BLAIRCRAFT 2100 ATTACK FIGHTER
6.0          characteristic length (b)
0.5          Mach
1.00         reduced frequency wb/U
s            s: symmetric a: anti-symmetric n: no symmetry
 0.0  0.0    x and y coord of inboard  leading  edge
12.0  0.0    x and y coord of inboard  trailing edge
12.0 12.0    x and y coord of outboard trailing edge
 0.0 12.0    x and y coord of outboard leading  edge
3            number of chordwise cuts (discretized x)
3            number of spanwise cuts (discretized y)

COMMENTS:

Pdimensional pressure
Cpnon-dimensional pressure
rhoair density

P = rho*U^2*Cp

For typical "non-dimensional" input, set U=b=1.
The output is interpreted accordingly.
Only Cp is printed out for each box.
```

bc_INPUT

```
Boundary condition (upwash) input for doublet lattice:

flag     constant     x power     y power
1        -1.0         0           0
0        -1.0         1           0
0        -1.0         0           1
```

```
0       -1.0          2          0
0       -1.0          1          1
0       -1.0          0          2
end of data
```

interpretation:
$w(x,y) = a00 + a10*x + a01*y + a20*x^2 + a11*x*y + a02*y^2$

instructions:

Only data with a "1" in the first column will be considered data.
Replace the "1" with a "0" to ignore any data.
A line which begins with an "e" will terminate the input.
There must be at least one line which begins with an "e".

end of data

APPENDIX C

The Doublet Lattice Program Output Listing

```
 Auxillary runtime data placed in file [dl.TRASH]

 Aerodynamic influence coefficients placed in file [dl.AIC]

 MAXBOX: 400 MAXDIM: 160000

Begin input
Input data will be read from file [dl.INPUT]
TITLE:
text: [BLAIRCRAFT 2100 ATTACK FIGHTER]
characteristic length: 6.000000
Mach: 0.500000
reduced frequency: 1.000000
Assume symmetry about the x axis
text: [ 0.0 0.0x and y coord of inboard leading edge]
inboard leading edge: x 0.000000 y 0.000000
inboard trailing edge: x 12.000000 y 0.000000
outboard trailing edge: x 12.000000 y 12.000000
outboard leading edge: x 0.000000 y 12.000000
number of boxes in the x direction: 3
number of boxes in the y direction: 3
Input complete

The wing area used to non-dimensionalize lift is: 4.0000e+00
The wing centroid is at x: 1.000000 and y: 1.000000
Begin discretizing the wing.
Discretization is now complete.
 reduced freq (k): 1.0000e+00
 beta^2 (1-M^2): 7.5000e-01

 0 1 2 3 4 5 6 7 8 0
 1 2 3 4 5 6 7 8
```

```
Boundary condition data will be read from file [dl.INPUT]
reduced frequency (k): 1.0000e+00
 [1] a: -1.000000 px: 0 py: 0

 compute the upwash at 9 control points

 Upwash specified and pressure will be computed
 0 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 1 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 2 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 3 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 4 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 5 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 6 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 7 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00
 8 Real[w]: 0.0000e+00 Imag[w]: -1.0000e+00

Solve the complex problem {w} = [D]{p}


Cp PRESSURE COEFFICIENTS (P=0.5*rho*U^2*Cp)
box # (real Cp) (imag Cp) (box area)
 0 -5.4900e-01 6.2682e+00 1.6000e+01
 1 -3.8862e+00 2.4495e+00 1.6000e+01
 2 -3.8736e+00 1.1745e+00 1.6000e+01
 3 -5.9144e-01 5.8092e+00 1.6000e+01
 4 -3.6405e+00 2.1530e+00 1.6000e+01
 5 -3.6234e+00 1.0281e+00 1.6000e+01
 6 -5.8286e-01 4.5474e+00 1.6000e+01
 7 -2.8983e+00 1.4663e+00 1.6000e+01
 8 -2.8893e+00 7.1186e-01 1.6000e+01

The characteristic length b: The non-dimensional wing area used
to non-dimensionalize lift is: 4.0000e+00
Lift Coefficient = C_L*q*A
THE COMPLEX WING LIFT COEFFICIENT (C_L) IS:
[(-2.5038e+00) + ( 2.8453e+00)i]
MAGNITUDE: ( 3.7901e+00) PHASE: ( 131.3471)deg
```

end of data

U.S. Government Printing Office 648-127