AIRFRAME AND EQUIPMENT ENGINEERING

REPORT NO. 43*

OUTLINE OF AN ACCEPTABLE METHOD OF VIBRATION AND FLUTTER ANALYSIS FOR A CONVENTIONAL AIRPLANE

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Prepared by: R. H. Scanlan
R. H. Scanlan

Robert Rosenbaum
R. Rosenbaum

Approved by: A. A. Vollmecke
A. A. Vollmecke
Chief, Airframe and
Equipment Engineering
Division

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AVIATION SAFETY RELEASE NO. 302

SUBJECT: Airframe and Equipment Engineering Report No. 43, "Outline of an Acceptable Method of Vibration and Flutter Analysis for a Conventional Airplane"

SUPERSEDES: Certificate and Inspection Division Release No. 37
Certificate and Inspection Division Release No. 46
Safety Regulation Release No. 68

The purpose of this release is to transmit to the Aircraft Industry a new, simplified, tabular method of vibration and flutter analysis, for use by relatively inexperienced personnel. It represents acceptable and recommended practice but is not intended as required procedure to meet the flutter prevention requirements in the Civil Air Regulations.

The subject report supersedes:

1. Aircraft Airworthiness Section Report No. 22
"Flexure - Torsion Binary Flutter" by Jean Wylie
February 1941.

2. Aircraft Airworthiness Section Report No. 23
"Perpendicular Axes Control Surface Binary Flutter"
by Jean Wylie April 1941.

3. Engineering Section Report No. 24
"Parallel Axes Control Surface Binary Flutter"
by Jean Wylie September 1941.

These reports are now considered to be obsolete and should therefore no longer be used to substantiate freedom from flutter.

A. S. Koch
Assistant Administrator
for Aviation Safety

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OUTLINE OF AN ACCEPTABLE METHOD OF VIBRATION AND FLUTTER ANALYSIS FOR A CONVENTIONAL AIRPLANE

Purpose

The object of this Report is to present the minimum of necessary information and technique on three-dimensional flutter calculation to cover a conventional case. It is intended to be used primarily by those engineers previously unacquainted with flutter problems who may be faced with the problem of meeting anti-flutter requirements during design of an aircraft to fly at "incompressible" air speeds. It is intended to be used as a simple introduction to engineering calculation which should be supplemented by use of AAF TR 4798, "Application of Three-Dimensional Flutter Theory to Aircraft Structures."

Appendices II to V of this Report are intended as additional material which covers some of the more important standard background of the usual flutter engineer. However, this material is placed in Appendices rather than the main body of the Report because the authors of the Report consider the Appendix material unnecessary to those requiring only a bare minimum in the way of flutter analysis.

Summary

The basic Report presents a brief introduction to the concepts of vibration and flutter calculations on a conventional aircraft wing. Just the necessary geometric conventions are established; then mass properties of wing and aileron are described. The next section gives a description, with examples, of how to calculate uncoupled vibration modes which are used in the ensuing analysis. The main section of the Report describes in detail, step-by-step use of a tabular technique for making the three-dimensional wing flutter calculations required. A complete illustrative example is included. The basic theory of flutter is not considered in the Report, but rather the emphasis is placed on solution by means of the given tabular technique.

This technique requires the use of a computing machine having an automatic multiplication feature. It is not claimed that the technique is exceptional nor preferable to others employed by already-skilled flutter analysis. It is merely presented as one proved means of meeting the minimum calculation requirements conventionally encountered. The technique is three-dimensional, i.e., includes the spanwise effect of wing modal patterns during vibration, and is thus far superior to two-dimensional calculations, which are now generally considered obsolete.

The final section of the main body of the Report deals briefly with minimum considerations on empennage vibration and flutter calculations. The means employed in this discussion is to relate the empennage problem directly to what has been previously developed in detail for the wing problem.

The appendices cover in detail the following more "advanced" topics:

II. The solution of frequency equations by matrix technique (including the obtaining of modes higher than the fundamental)

III. Coupled modes of vibration of a free-free wing in air (using matrix technique)

IV. Three-degree, three-dimensional flutter theory (standard theory logically developed and presented)

V. Wing flutter calculation based on coupled vibration modes (theory of using ground vibration modes rather than uncoupled modes directly in flutter analysis)

Scope

The aim of the Report being to present only a minimum flutter analysis, the scope of material herein is necessarily restricted. However, the analysis as presented aims at omitting no essential consideration, such as, relative to vibration; describing the minimum calculation technique needed to get uncoupled vibration modes for subsequent flutter calculation, including symmetric and unsymmetric bending and torsion modes; relative to flutter, employing at least three degree of freedom, including the movable control surface; employing spanwise (three-dimensional) modal effects on mechanical and aerodynamic terms; using the actual parameters of the airplane in question rather than employing oversimplifying assumptions; taking into account the effect of taper of the fixed surface on air force expressions.

Certain considerations are however, omitted. Among these are, relative to vibration; the use of modes higher than the fundamental; the calculation or test measurement of coupled modes; relative to flutter; the employment of more than three degree of freedom; the inclusion of effects of wing taper over the region of the control surface; the twist of the control surface; any effects of compressibility; effects of free-body motion of the entire airplane; the effect of aspect ratio on oscillatory air forces.

The scope is limited to application to conventional airplanes, and of these, usual cases only. It is entirely possible that unusual circumstances, even on conventional airplanes, will require more elaborate analyses than herein presented. However, the authors believe that the design parameters available for variation, in particular control surface balance weights will often be adjusted to within reasonable values by the minimum analysis suggested herein and more elaborate analyses are seldom warranted.

While the analysis considers "incompressible" air only, it is considered that it can be applied to cover cases in a range of Mach numbers up to .8 in cases where a very substantial flutter safety margin is calculated.

A. WING GEOMETRIC PROPERTIES

1. Breakdown into strips

The first step in a vibration and flutter analysis of a conventional airplane wing is to assume it broken down into a number of chordwise strips, as illustrated in Fig. 1.

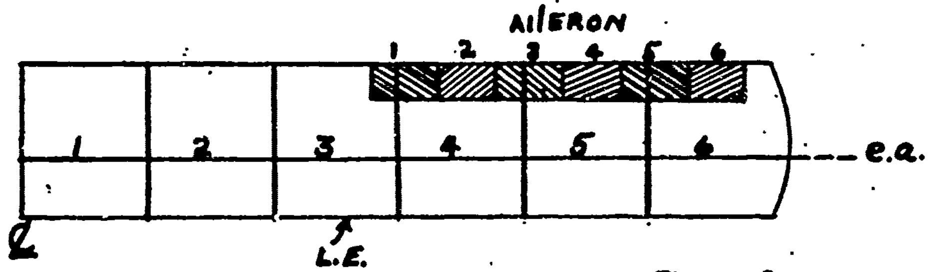

Figure 1

The first strip has its inboard end at the centerline of the airplane.

The aileron is also assumed broken into chordwise strips in similar fashion.

2. Elastic Axis

The elastic axis can be defined as a line, fixed within the wing, about which the wing twists when under torsional loads. (Actually such a line may not be exactly determinable with assurance of accuracy or fixity since it may vary with type of wing loading and other factors. However, experience has shown that a reasonable assumption of the position of this axis can be made and that such an assumption will lead to acceptable results in flutter calculation.)

If the airplane is in the design stage the axis may be taken as the locus of the shear centers of wing torque boxes or, in the case of two-spar wing, a line through those points which are located between the spars in inverse distance from the spars as the spar section moments of inertia.

If the airplane is available for test, a test procedure is provided for example, by the use of two jacks mounted on platform scales or otherwise equipped to measure their loads. At the center chord of each wing strip a forward and an aft location (for example, front and rear spars) may be chosen. With the jacks located symmetrically on right and left wings, loads of the same magnitude are applied by each jack on a forward push-up point at a given strip. The deflections $(d_f)_f$ and $(d_r)_f$, respectively of front and rear spars are noted with respect to a reference plane determined by two distince lines intersecting and perpendicular to the airplane centerline and parallel to the ground. The procedure is repeated with the same load and the jacks at the aft push-up point of the same chord. The corresponding deflections $(d_f)_r$ and $(d_r)_r$ are measured with respect to the same reference plane. Let R be the distance between the reference points and r the distance from the forward point to the elastic axis. Let $d_{e.a}$ be the deflection of the elastic axis. Then for a load at the forward point:

$$d_{e.a} = (d_r)_f + \frac{(d_f)_f - (d_r)_f}{R} \times (R-r);$$

and for the same load at the rearward point of the same section:

$$d_{e.a} = (d_f)_r + \frac{(d_r)_r - (d_f)_r}{R} \times r$$

Equating these two values of elastic axis deflections, which must be the same for the same applied load, there is obtained an expression which may be solved for r, thus locating the elastic axis at the section in question. The procedure may be repeated for each strip into which the wing is divided.

Once the elastic axis is determined for the entire wing it is convenient to replace it, if it is slightly curved, by a straight line faired through it at a constant percentage of chord. This makes for considerable ease of calculation subsequently.

3. Notation

The following conventions will be established (see Fig. 2) for each stripwise section:

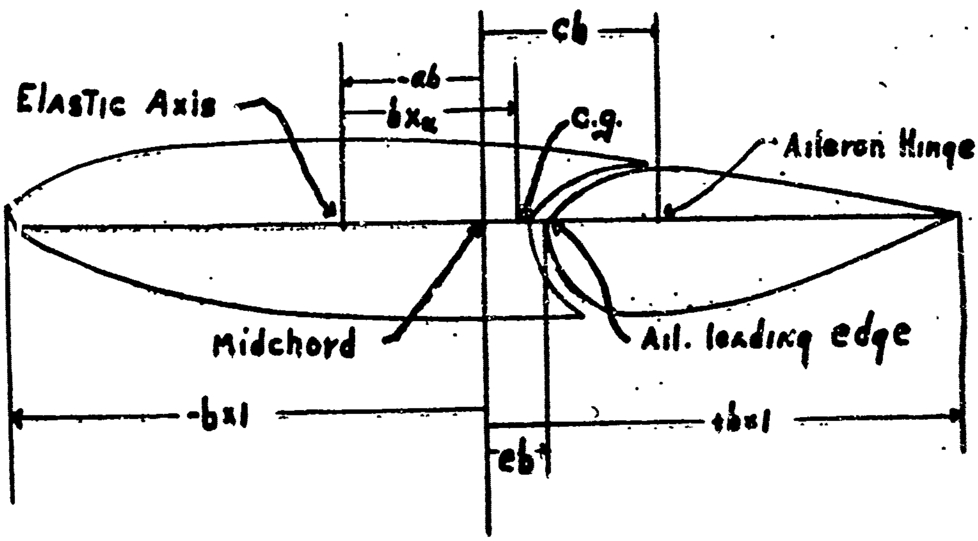

Fig. 2

b = semi-chord (ft.) of wing at section in question
eb = distance (ft.) from wing midchord to aileron leading edge
cb = distance (ft.) from wing midchord to aileron hinge line
$x_\alpha b$ = distance (ft.) from wing elastic axis to c.g. of section
ab = distance (ft.) from wing midchord to elastic axis of section

In all cases measurements are positive aft from the wing midchord, or, in the case of x_α , aft from the elastic axis; and they are negative in the opposite direction. The above quantities are determined for each strip of the wing.

As mentioned for the elastic axis, it is convenient for subsequent calculations to fair a straight line through any slightly curved line in the wing planform representing the locus of points eb, ab, or cb at each section, thus establishing a constant value for e, a, and c for the subsequent flutter analysis.

For purpose developed later, a representative semi-chord b_r (feet for the entire wing is needed. This may be taken as; average semi-chord of the wing, the semi-chord at the 75% semi-span station, or the semi-chord at the aileron midspan section, whichever appeals to the analyst as most suitable for this particular problem.

B. WING MASS PROPERTIES

The following properties are calculated for each wing strip: (including any aileron present in the strip)

1. Total mass (slugs $= \frac{\text{lbs.})}{32.2}$: m

2. Total static moment (slugs x ft.) about the elastic axis of the given strip:

$$S_\alpha = m x_\alpha b$$

3. Total mass moment of inertia (slugs x ft.^{2}) about the elastic axis of the given strip:

$$I_\alpha = I_{cg} + m(x_\alpha b)^2$$

where I_{cg} is the total mass moment of inertia of the strip about a span-wise axis parallel to the elastic axis of the strip and passing through the center of gravity of the strip:

C. AILERON MASS PROPERTIES

The following properties are calculated for each aileron strip:

1. Total static moment (slugs x ft.) of the given strip about the aileron hinge line:

$$S_\beta$$

2. Total mass moment of inertia (slugs x ft.^{2}) of the given strip about the aileron hinge line:

$$I_\beta$$

3. Product of inertia (slugs x ft.^{2}) of the given strip about the aileron hinge line and wing elastic axis:

$$P_{\alpha\beta} = S_\beta (c-a)b + I_\beta$$

D. Wing uncoupled vibration modes

1. Wing influence coefficients

If the airplane is in the design stage the influence coefficients may be calculated from stress data; if it is completed they may be measured in a simple test.

(a) Torsional influence coefficients (x.y). These are defined as the torsional deflection θ in radians at a spanwise strip x due to a torque of one foot pound applied in the plane of spanwise strip y. The exact method of analytical determination of these coefficients depends on the structure of the particular wing under consideration. If the airplane is available for test, however, the influence coefficients may be determined directly by a method similar to the push-up method described for determining the elastic axis. This method consists simply of determining the deflection (relative to the wing root) of each "push-up" point for a given unit load successively at each other "push-up" point. The torsional deflections under a unit torque at a given station can then be calculated from theoretical applications, at a given section, of equal and opposite loads of properly scaled magnitude. The set of influence coefficients in torsion can be arrayed in a tabular or matrix form as illustrated:

TYPICAL MATRIX OF TORSION INFLUENCE COEFFICIENTS θ (x.y) x10^9 (radians/ft.lb)

12.00	28.56	42.00	51.96	51.96	51.96
28.56	48.00	121.[illegible]	138.00	138.00	138.00
42.00	121.56	195.00	246.00	246.00	246.00
51.96	138.00	246.00	369.00	405.00	405.00
51.96	138.00	246.00	405.00	798.00	798.00
51.96	138.00	246.00	405.00	798.00	1776.00

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(b) Bending influence coefficients $\delta(x.y)$. These are defined as the vertical deflection of the elastic axis at spanwise station x due to application of one pound vertical load at spanwise station y. The set of influence coefficients in bending can be arrayed in a matrix as illustrated below:

TYPICAL MATRIX OF BENDING INFLUENCE COEFFICIENTS

$\delta(x.y) \times 10^7$ (ft./lb.)

.36	1.96	3.27	5.07	7.53	9.72
1.96	6.50	12.50	21.17	32.83	45.00
3.27	12.50	25.67	46.00	73.50	102.83
5.07	21.17	46.00	92.17	157.75	228.08
7.53	32.83	73.50	157.75	303.08	470.42
9.72	45.00	102.83	228.08	470.42	817.17

(c) Symmetry of influence coefficients It will be noted that both torsion and bending influence coefficient matrices are symmetrical about their principal diagonals (items listed from upper left to lower right corner). This is due to the nature of elastic structures as expressed by Maxwells Law of Reciprocal Deflections. (Note: The above matrices are chosen arbitrarily and are not associated with the example employed later in flutter analysis.)

2. Uncoupled torsional vibration mode.

When a wing is restrained against bending it can vibrate only in its natural torsional modes. These are defined as the natural simple harmonic vibratory configurations or shapes which a wing assumes when vibrating in pure torsion only. (Such pure vibration is impossible in actual practice, where always some coupling occurs between bending and torsion. However, for convenient calculation purposes which appear later, uncoupled modes are used.)

For calculation of a vibration mode the material required consists of (1) the set of torsional influence coefficients and (2) the strip-wise values of I_α. The equations of simple harmonic motion can then be expressed as [Torsional deflection at station i] = [Sum of {inertia torques} x {influence coefficients affecting station i}]

In algebraic symbols

$$\alpha_1 = T_1\,\Theta(1,1) + T_2\,\Theta(1,2) + \cdots\cdots\; T_6\,\Theta(1,6)$$

$$\alpha_2 = T_1\,\Theta(2,1) + T_2\,\Theta(2,2) + \cdots\cdots\; T_6\,\Theta(2,6)$$

$$\vdots$$

$$\alpha_6 = T_1\,\Theta(6,1) + T_2\,\Theta(6,2) + \cdots\cdots\; T_6\,\Theta(6,6)$$

where T_i is the inertia torque at spanwise strip i:

$$T_i = \left[\frac{f_\alpha \cdot 2\pi}{60}\right]^2 I_{\alpha i}\,\alpha_i$$

where f_α is the torsional vibrating frequency in cycles per minute, $I_{\alpha i}$ is the mass moment of inertia of strip i (slugs ft^2) about the elastic axis and α_i is the angular deflection of strip i in radians. The equations of motion then become

$$\alpha_1 = \left[\frac{2\pi f_\alpha}{60}\right]^2 \left[I_{\alpha 1}\Theta(1,1)\alpha_1 + I_{\alpha 2}\Theta(1,2)\alpha_2 + \cdots I_{\alpha 6}\Theta(1,6)\alpha_6\right]$$

$$\vdots$$

$$\alpha_6 = \left[\frac{2\pi f_\alpha}{60}\right]^2 \left[I_{\alpha 1}\Theta(6,1)\alpha_1 + I_{\alpha 2}\Theta(6,2)\alpha_2 + \cdots I_{\alpha 6}\Theta(6,6)\alpha_6\right]$$

These equations can be solved by the much-used method of <u>iteration</u>. This is done by dealing with the <u>coefficients</u> $I_{\alpha j}\ \Theta(i,j)$ <u>only</u> instead of the entire equation. They are arranged in tabular or matrix form in just the same positions they would occupy in the ordinary algebraic equations above, but algebraic signs are omitted. The result is the following <u>matrix equation</u>:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \left(\frac{2\pi f_\alpha}{60}\right)^2 \begin{bmatrix} I_{\alpha_1}\theta(1,1) & I_{\alpha_2}\theta(1,2) & \cdots & I_{\alpha_6}\theta(1,6) \\ \vdots & & & \vdots \\ I_{\alpha_1}\theta(6,1) & \cdots & \cdots & I_{\alpha_6}\theta(6,6) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$

This equation is equivalent to the previous algebraic equations when "row by column" multiplication is used, i.e., a row of the square array or table is matched item by item with the last column of α's and multiplied termwise; then the results are added together.

As an example, assume that a set of $I_{\alpha i}$ values are known:

$$\begin{aligned} I_{\alpha_1} &= 30.458 \text{ slugs ft}^2 \\ I_{\alpha_2} &= 16.478 \\ I_{\alpha_3} &= 13.114 \\ I_{\alpha_4} &= 9.972 \\ I_{\alpha_5} &= 5.283 \\ I_{\alpha_6} &= 2.568 \end{aligned}$$

The matrix of products $I_{\alpha i} \cdot \theta(i,j)$ is expressed below (using the arbitrary matrix of torsional influence coefficients developed above): This result is called the dynamic matrix for uncoupled torsion:

$$\begin{bmatrix} 365.49 & 470.62 & 550.80 & 518.13 & 274.48 & 133.45 \\ 869.87 & 790.96 & 1594.16 & 1376.09 & 728.99 & 354.43 \\ 1279.22 & 2003.11 & 2596.61 & 2453.03 & 1299.50 & 631.81 \\ 1582.57 & 2274.01 & 3226.09 & 3679.55 & 2139.41 & 1040.18 \\ 1582.57 & 2274.01 & 3226.09 & 4038.53 & 4215.44 & 2049.53 \\ 1582.57 & 2274.01 & 3226.09 & 4038.53 & 4215.44 & 4561.36 \end{bmatrix} \times 10^{-9}$$

Above, all the numbers of the first column are the product of 30.458 times the respective elements of the first column of the matrix of influence coefficients; all the numbers of the second column are products of 16.478 times the respective elements of the second column of the influence coefficient matrix; etc.

The solution of the dynamic matrix equation proceeds by iteration, as follows, on the dynamic matrix: Assume any column of six numbers, the largest of which (in the last place) is 1:

.0868
.2255
.4029
.5853
.7870
1.0000

(This column was assumed on the basis of some prior knowledge as to the expected mode shape. This knowledge is a help but is not essential.) Multiply the dynamic matrix "row-by-column" by this column, obtaining the following result:

1012
2630
4699
6828
9681
12193

Divide all the numbers of this column by 12193, obtaining 1 as the largest quotient; this is called <u>normalizing</u>:

.0830
.2157
.3854
.5600
.7940
1.0000

Repeat the above process until the normalized row <u>converges</u>

(i.e., gives the same result to a desired degree of accuracy on two successive iterations). The complete results are given below:

	Assumed Mode				Final Mode
	.0868	.0830	.0819	.0816	.0815
	.2255	.2157	.2128	.2121	.2119
	.4029	.3854	.3802	.3789	.3786
	.5853	.5600	.5538	.5521	.5517
	.7870	.7940	.7913	.7902	.7899
	1.0000	1.0000	1.0000	1.0000	1.0000
Divisor $\dfrac{10^9}{\left(\dfrac{2\pi f_\alpha}{60}\right)^2} =$		12193	12035	11974	11956

2/5/6

It is seen that the last two normalized columns are identical to the third decimal place, and the resulting mode in uncoupled torsion is

Sta 1	.082
Sta 2	.212
Sta 3	.379
Sta 4	.552
Sta 5	.790
Sta 6	1.000

The frequency associated with this mode is given from the relation

$$\left(\frac{60}{2\pi f_\alpha}\right)^2 \times 10^9 = 11956$$

or

$$f_\alpha = 2762 \quad \text{cpm}$$

The above fundamental torsional mode and frequency have been calculated on the assumption that the fuselage mass pitching moment of inertia is infinitely large, (i.e., the fuselage is immobile). This is a reasonable assumption in most analyses.

3. Uncoupled bending vibration modes.

This procedure is very similar to the calculation of torsional modes. The iteration process employed is identical. However, the mass of the fuselage enters these calculations and, as a result, there must be expressed means of calculating two types of bending modes; symmetrical and unsymmetrical.

In the symmetrical modes of bending the airplane and its wing take a modal shape which is identical on left and right sides, and may be represented schematically as in Fig. 3.

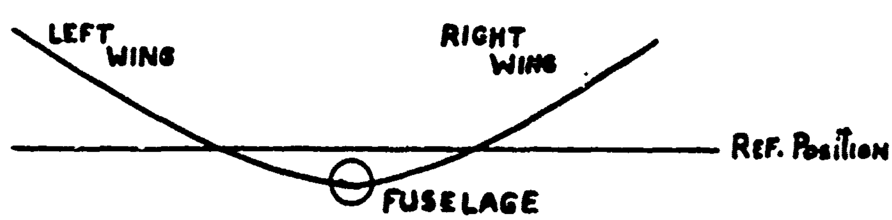

Figure 3

It can be seen in Fig. 3 that the fuselage translates vertically about its reference position in the symmetrical mode.

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In the unsymmetrical modes of bending the airplane and its wing take a modal shape in which the position of the left side is "mirrored" in the right side, i.e., the same deflection occurs but is negative in direction on the opposite side. This is illustrated in Figure 4.

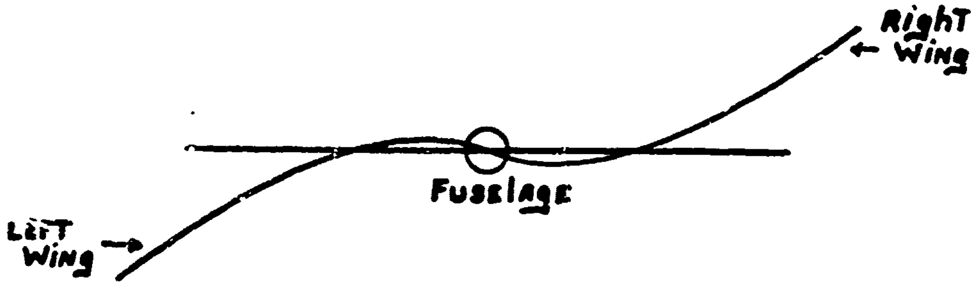

Figure 4

(a) Symmetrical mode calculation. The expression for determining that a mode be symmetrical is that the sum of vertical inertia forces be zero on one side of the airplane. Since inertia forces in harmonic vibration are the products of mass times acceleration, and acceleration is proportional to deflection, there is obtained the expression.

$$m_0 h_0 + m_1 h_1 + m_2 h_2 + \text{———} + m_6 h_6 = 0$$

where m_i is the mass (slugs) of the i^{th} wing strip (including half the weight of the fuselage in the "zero" strip at the airplane center line) and h_i is the deflection (feet) of the i^{th} strip.

The equations of motion state that deflection at strip i is the sum of inertia forces at the various strips multiplied by the bending influence coefficients which affect strip i; deflection will now be measured with respect to the deflection of the "zero" strip, or fuselage centerline:

$$h_1 - h_0 = F_1\delta(1,1) + F_2\delta(1,2) + \cdots + F_6\delta(1,6)$$
$$h_2 - h_0 = F_1\delta(2,1) + F_2\delta(2,2) + \cdots + F_6\delta(2,6)$$
$$\vdots$$
$$h_6 - h_0 = F_1\delta(6,1) + \cdots + F_6\delta(6,6)$$

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Where $F_i = \left(\frac{2\pi f_h}{60}\right)^2 m_i h_i$ is the inertia force and $(\delta(i,j))$ is the influence coefficient representing bending deflection in feet (with respect to fuselage) of spanwise point i due to a one pound load at point j.

Thus the equations become

$$h_1 - h_0 = \left(\frac{2\pi f_h}{60}\right)^2 \left[m_1 \delta(1,1) h_1 + \cdots\cdots + m_6 \delta(1,6) h_6\right.$$

$$\vdots$$

$$h_6 - h_0 = \left(\frac{2\pi f_h}{60}\right)^2 \left[m_1 \delta(6,1) h_1 + \cdots\cdots + m_6 \delta(6,6) h_6\right.$$

The h_0 term must be removed before further calculation is feasible. This is accomplished by multiplying the first equation through by m_1, the second by m_2, etc., and adding the results. This gives

$$m_1 h_1 + m_2 h_2 \cdots + m_6 h_6 - (m_1 + m_2 + \cdots + m_6)\, h_0$$

$$= \left(\frac{2\pi f_h}{60}\right)^2 \left\{\left[m_1^2 \delta(1,1) + m_1 m_2 \delta(2,1) + m_1 m_3 \delta(3,1) \cdots\cdots + m_1 m_6 \delta(6,1)\right] h_1\right.$$

$$\left. + \cdots\cdots + \left[m_1 m_6 \delta(1,6) + m_2 m_6 \delta(2,6) + \cdots\cdots m_6^2 \delta(6,6)\right] h_6 \right\}$$

But from the condition for symmetrical modes:

$$m_1 h_1 + m_2 h_2 + \cdots + m_6 h_6 = -m_0 h_0$$

Substituting this in the above gives h_0 in terms of $\left(\frac{2\pi f_h}{60}\right)^2$

times a combination of h_1 to h_6 terms Replacing this in the original equation removes the h_0 term from each.

The above process will be made clearer by an illustrative example. Assume the previously given set of bending influence coefficients.

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Assume the following set of masses at wing strips:

m_0	87.20 slugs
m_1	7.99
m_2	8.15
m_3	10.00
m_4	4.26
m_5	2.05
m_6	1.52

The dynamic matrix for symmetric bending then consists of the coefficients in the following dynamic equations:

$$(h_1 - h_o) \times 10^7 = (2.88\, h_1 + 15.97\, h_2 + 32.70\, h_3 + 21.60\, h_4 + 15.44\, h_5 + 14.77\, h_6)\left(\frac{2\pi f_h}{60}\right)^2$$

$$(h_2 - h_o) \times 10^7 = (15.66\, h_1 + 52.98\, h_2 + 125.00\, h_3 + 90.18\, h_4 + 67.39\, h_5 + 68.40\, h_6)\left(\frac{2\pi f_h}{60}\right)^2$$

$$(h_3 - h_o) \times 10^7 = (26.13\, h_1 + 101.88\, h_2 + 256.70\, h_3 + 195.96\, h_4 + 150.68\, h_5 + 156.30\, h_6)\left(\frac{2\pi f_h}{60}\right)^2$$

$$(h_4 - h_o) \times 10^7 = (40.51\, h_1 + 172.54\, h_2 + 460.00\, h_3 + 392.64\, h_4 + 323.39\, h_5 + 346.68\, h_6)\left(\frac{2\pi f_h}{60}\right)^2$$

$$(h_5 - h_o) \times 10^7 = (60.16\, h_1 + 267.56\, h_2 + 735.00\, h_3 + 672.02\, h_4 + 621.31\, h_5 + 715.04\, h_6)\left(\frac{2\pi f_h}{60}\right)^2$$

$$(h_6 - h_o) \times 10^7 = (77.66\, h_1 + 366.75\, h_2 + 1028.30\, h_3 + 971.62\, h_4 + 967.36\, h_5 + 1242.10\, h_6)\left(\frac{2\pi f_h}{60}\right)^2$$

The equations are then successively multiplied by m_1, m_2, etc., and added. The result is:

$$-33.97h_o + \sum_1^6 m_i h_i = (825.88h_1 + 3419.17h_2 + 8876.39h_3 + 7394.30h_4 + 6295.81h_5 + 7069.15h_6)\left(\frac{2\pi f_h}{60}\right)^2 10^{-7}$$

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or, since

$$m_1h_1 + m_2h_2 + m_3h_3 + m_4h_4 + m_5h_5 + m_6h_6 = -m_0h_0 = -87.20\ h_0$$

h_0 can be expressed as:

$$h_0 = -\left(\frac{2\pi f_h}{60}\right)^2 \times 10^{-7} \left[6.82h_1 + 28.22h_2 + 75.26h_3 + 61.02h_4 + 51.96h_5 + 58.\text{[illegible]}h_6 \right]$$

Thus h_0 can be eliminated from the dynamic equations. The result in matrix form is

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} = 10^{-7} \begin{bmatrix} -3.94 & -12.25 & -40.56 & -39.42 & -36.52 & -43.57 \\ 8.84 & 24.76 & 51.74 & 29.16 & 15.34 & 10.06 \\ 19.31 & 73.66 & 183.44 & 134.94 & 98.72 & 97.96 \\ 33.69 & 144.32 & 386.74 & 331.62 & 271.43 & 288.34 \\ 53.34 & 239.34 & 661.74 & 611.00 & 569.35 & 656.70 \\ 70.84 & 338.53 & 955.04 & 910.60 & 912.40 & 1183.76 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} \left(\frac{2\pi f_h}{60}\right)^2$$

Iteration on this gives the following results:

<u>Assumed Mode</u>

	-.04	-.039	-.039	-.039
	-.02	.016	.016	.016
	.11	.103	.103	.103
	.30	.281	.280	.280
	.61	.602	.594	.594
	1.00	1.000	1.000	1.000
$\dfrac{10^7}{\left(\frac{2\pi f_h}{60}\right)^2} =$		2133	2094	2085

The frequency f_h is defined by

$$\frac{10^7}{\left(\frac{2\pi f_h}{60}\right)^2} = 2085$$

or $f_h = 661$ cpm

Using the previous expression for h_0 in terms of $h_1, \ldots h_6$, the value of h_0 is found:

$$h_0 = -.055$$

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Then the following is the complete definition of the first symmetrical bending mode:

Sta. 0	-.055
Sta. 1	-.039
Sta. 2	.016
Sta. 3	.103
Sta. 4	.280
Sta. 5	.594
Sta. 6	1.000

Note that this mode accounts for a displacement of the fuselage in opposite phase to that of the wing tip, as shown in Figure 3.

(b) Unsymmetrical mode calculation. The expression for determining that a mode be unsymmetrical is that the sum of rolling inertia moments be zero:

$$m_1 h_1 y_1 + m_2 h_2 y_2 + \cdots\cdots\cdots m_6 h_6 y_6 + I_f \Theta = 0$$

where h_i is the displacement (feet) of strip i from its equilibrium position, y_i is the distance (feet) from the airplane centerline spanwise to the center of strip i, I_f is the fuselage mass moment of inertia in roll about its centerline, Θ is the angular roll of the fuselage during vibration. The equations of motion in this case are

$$h_1 - y_1 \Theta = \left(\frac{2\pi f_h}{60}\right)^2 \left[m_1 h_1 \delta(1,1) + \cdots\cdots\cdots m_6 h_6 \delta(1,6) \right.$$

$$\vdots$$

$$h_6 - y_6 \Theta = \left(\frac{2\pi f_h}{60}\right)^2 \left[m_1 h_1 \delta(6,1) + \cdots\cdots\cdots m_6 h_6 \delta(6,6) \right.$$

To eliminate the terms in Θ , a procedure analagous to that used in the symmetric case is employed. Multiply the first equation by $m_1 y_1$, the second by $m_2 y_2$, etc., and add results; thus

$$m_1 y_1 h_1 + m_2 y_2 h_2 + m_3 y_3 h_3 + m_4 y_4 h_4 + m_5 y_5 h_5 + m_6 y_6 h_6 - (m_1 y_1^2$$
$$+ m_2 y_2^2 \cdots + m_6 y_6^2)\Theta = \left(\frac{2\pi f_h}{60}\right)^2 \{ [m_1^2 y_1 \delta(1,1) + m_1 m_2 y_2 \delta(2,1) + \cdots$$
$$+ m_1 m_6 y_6 \delta(6,1)] h_1 + \cdots\cdots + [m_1 m_6 y_1 \delta(1,6) + m_6 m_2 y_2 \delta(2,6)$$
$$+ \cdots\cdots + m_6^2 y_6 \delta(6,6)] h_6$$

But from the equilibrium condition for unsymmetric modes

$$m_1 y_1 h_1 + \cdots\cdots\cdots\cdots \; m_6 y_6 h_6 = - I_f \theta$$

Hence the left hand of the above equation becomes

$$-\theta\left[m_1 y_1^2 + m_2 y_2^2 + \cdots\cdots\cdots \; m_6 y_6^2 + I_f\right]$$

which can be recognized as $-\theta I_a$

where I_a is rolling moment of inertia of ½ of the entire airplane.

Thus θ can be expressed as $\left(\frac{2\pi f_h}{60}\right)^2$ times a combination of the h_i terms, and θ can thus be eliminated. An illustrative example will be presented to make this point clear. The first unsymmetric mode will be worked out for the airplane having the previously given set of influence coefficients in bending. The dynamic equations for unsymmetric bending are the same as those for symmetric bending when $h_i - h_o$ is replaced by $h_i - y_i\theta$. In matrix form these are

$$\begin{bmatrix} h_1-y_1\theta \\ h_2-y_2\theta \\ h_3-y_3\theta \\ h_4-y_4\theta \\ h_5-y_5\theta \\ h_6-y_6\theta \end{bmatrix} = \begin{bmatrix} 2.88 & 15.97 & 32.70 & 21.60 & 15.44 & 14.77 \\ 15.66 & 52.98 & 125.00 & 90.18 & 67.30 & 68.40 \\ 26.13 & 101.88 & 256.70 & 195.96 & 150.68 & 156.30 \\ 40.51 & 172.54 & 460.00 & 392.64 & 323.39 & 346.68 \\ 60.16 & 267.56 & 735.00 & 672.02 & 621.31 & 715.04 \\ 77.66 & 366.75 & 1028.30 & 971.62 & 964.36 & 1242.10 \end{bmatrix} \times \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} 10^{-7}\left(\frac{2\pi f_h}{60}\right)^2$$

To eliminate the terms in θ the values of y_i are needed.

In this example they will be taken as

$$\begin{aligned} y_1 &= 2.950 \text{ feet} \\ y_2 &= 4.842 \\ y_3 &= 6.542 \\ y_4 &= 9.208 \\ y_5 &= 12.867 \\ y_6 &= 16.833 \end{aligned}$$

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Thus

m_1y_1 = 23.571 slug-ft.

m_2y_2 = 39.462

m_3y_3 = 65.420

m_4y_4 = 39.226

m_5y_5 = 26.377

m_6y_6 = 25.586

Multiplying the first dynamic equation by m_1y_1, the second by m_2y_2, etc., and adding results yields

$$-I_a\Theta = \left[7558h_1 + 32341h_2 + 86238h_3 + 74875h_4 + 66625h_5 + 77512h_6\right] 10^{-7} \left(\frac{2\pi f_h}{60}\right)^2$$

If I_a is taken as 21000 slug-ft:

$$\Theta = -\left[3.60h_1 + 15.40h_2 + 41.07h_3 + 35.65h_4 + 31.73h_5 + 36.91h_6\right] \times 10^{-7} \left(\frac{2\pi f_h}{60}\right)^2$$

Hence

$$y_1\Theta = -\left(\frac{2\pi f_h}{60}\right)\left[10.62h_1 + 45.43h_2 + 121.16h_3 + 105.17h_4 + 93.60h_5 + 108.88h_6\right] 10^{-7}$$
$$y_2\Theta = -\left(\frac{2\pi f_h}{60}\right)\left[17.43h_1 + 74.57h_2 + 198.86h_3 + 172.62h_4 + 153.64h_5 + 178.72h_6\right] 10^{-7}$$
$$y_3\Theta = -\left(\frac{2\pi f_h}{60}\right)\left[23.55h_1 + 100.75h_2 + 268.68h_3 + 233.22h_4 + 207.55h_5 + 241.47h_6\right] 10^{-7}$$
$$y_4\Theta = -\left(\frac{2\pi f_h}{60}\right)\left[33.15h_1 + 141.80h_2 + 378.17h_3 + 328.27h_4 + 292.17h_5 + 339.87h_6\right] 10^{-7}$$
$$y_5\Theta = -\left(\frac{2\pi f_h}{60}\right)\left[46.32h_1 + 198.15h_2 + 528.45h_3 + 458.71h_4 + 408.27h_5 + 474.92h_6\right] 10^{-7}$$
$$y_6\Theta = -\left(\frac{2\pi f_h}{60}\right)\left[60.60h_1 + 259.23h_2 + 691.33h_3 + 600.10h_4 + 534.11h_5 + 621.31h_6\right] 10^{-7}$$

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The following matrix equation results from eliminating θ terms from the dynamic equations:

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} = \begin{bmatrix} -7.74 & -29.46 & -88.46 & -83.57 & -78.16 & -94.11 \\ -1.77 & -21.59 & -73.86 & -82.44 & -86.34 & -110.32 \\ 2.58 & 1.13 & -11.98 & -37.26 & -56.90 & -85.17 \\ 7.36 & 30.74 & 81.83 & 64.37 & 31.22 & 6.18 \\ 13.84 & 69.41 & 206.55 & 213.31 & 213.04 & 240.12 \\ 17.06 & 107.52 & 336.97 & 371.52 & 430.25 & 520.79 \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} 10^{-7}\left(\frac{2\pi f_h}{60}\right)^2$$

The solution of this matrix equation is obtained by the iteration process already used above. The results are below:

Assumed Mode

-.175	- 78.387	-.131	-98.762	-.1440	-103.865	-.1461
-.238	-103.525	-.173	-122.375	-.1784	-127.650	-.1796
-.237	-95.646	-.160	-102.680	-.1497	-105.454	-.1484
-.114	-19.285	-.032	-3.604	-.0053	-.418	-.0006
.296	210.969	.353	261.628	.3814	274.947	.3868
1.000	597.354	1.000	686.029	1.0000	710.836	1.0000

Result

-104.743	-.14646	-104.888	-.14651	-.147
-128.570	-.17977	-128.721	-.17981	-.180
-105.959	-.14815	-106.040	-.14812	-.148
.108	.00015	.196	.00027	.000
277.256	.38767	277.656	.38782	.388
715.178	1.00000	715.891	1.00000	1.000

$$f_h = \frac{60}{2\pi}\sqrt{\frac{10^7}{715.9}} = 1129 \text{ cpm}$$

The uncoupled symmetric bending, unsymmetric bending, and torsion modes are plotted in Figure 5.

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Note: The detailed discussion of calculation of fundamental and higher modes of any given matrix is given in Appendix II. The theory for calculation of coupled modes is given in Section X of this report and in Appendix III.

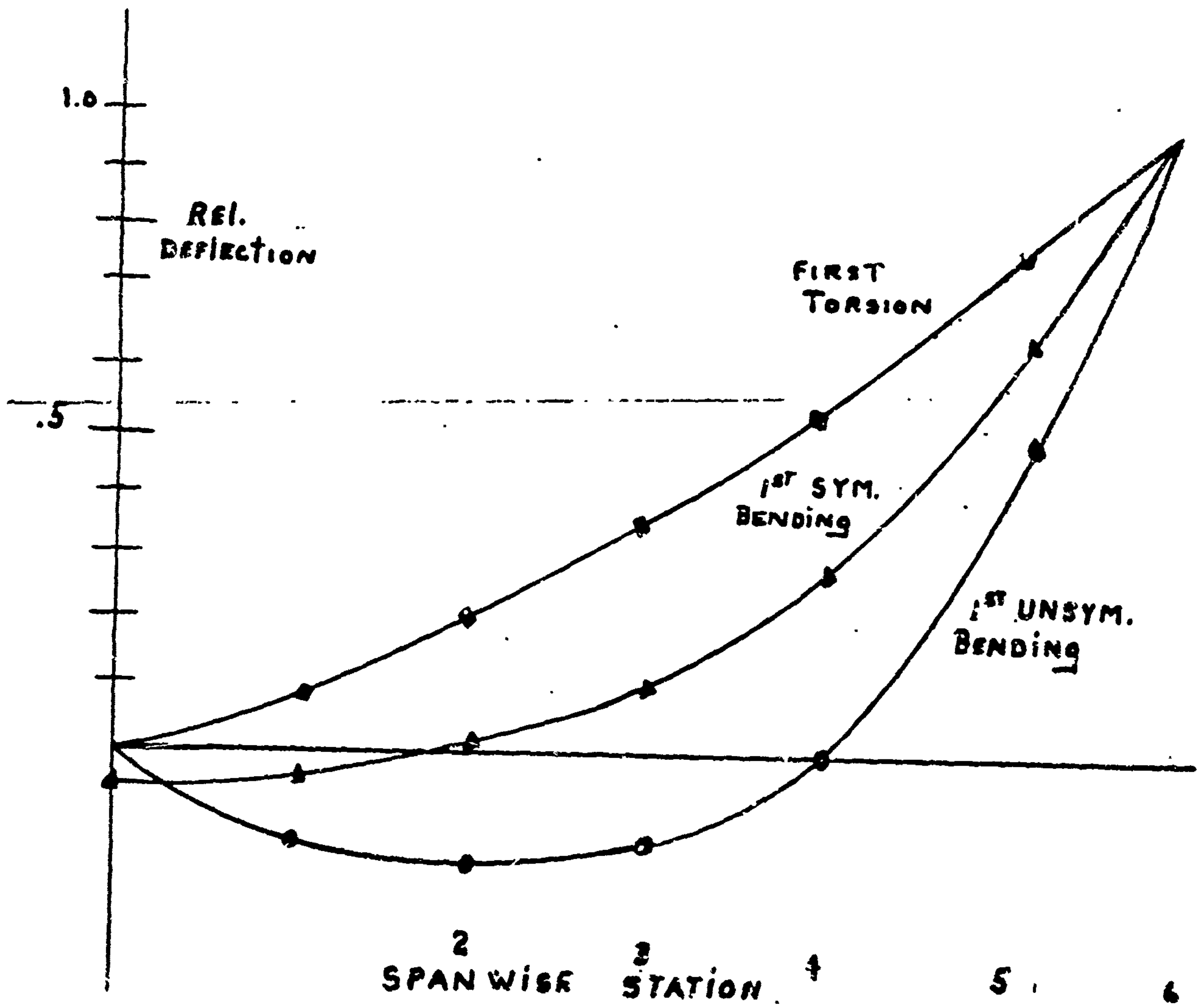

Figure 5

E. WING FLUTTER ANALYSIS

1. Theory

The theory upon which this section of the Report is based is found in Army Air Forces Technical Report No. 4798, "Application of Three-Dimensional Flutter Theory to Aircraft Structures," by Benjamin Smilg and Lee S. Wasserman. The reader is assumed to have a copy of this report available. It will be directly employed to furnish certain aerodynamic terms necessary in flutter calculations. The theoretical considerations discussed here will be confined to those of computation only, and not to the basic theory of flutter. The essential contribution of this section is the presentation of the computation scheme in a sequence of concise tables.

The flutter condition for a given airplane wing exists when that wing is flying at a certain critical airspeed. This critical airspeed is that which maintains definite harmonic oscillations of the airfoil in its natural configurations without allowing them to damp out. The phenomenon of flutter is classed with self-excited vibration phenomena and is characterized by the fact that the wing at the critical flutter speed picks up energy of motion from the air stream (by virtue of its position) as rapidly as it can dissipate it by internal damping or other means.

The mathematical solution of the problem assumes the wing to have three degrees of freedom in which to move at any section: vertical (bending), rotational (torsion) and the rotation of any attached flapped surface about its hinge line(aileron).

The solution is further denoted as three-dimensional (rather than two-dimensional) because the spanwise dimension of the wing enters the calculation through the modes (characteristic vibratory shapes) taken by the entire wing in bending and torsion. The air force effects calculated for each two-dimensional section of the wing are integrated over these modes spanwise to obtain a three-dimensional effect. Usually, the (relatively small) effects of wing aspect ratio are neglected for the usual analysis based upon incompressible flow.

The mathematical problem resolves itself into the solution of the following three-degree-of-freedom-stability determinant:

$$\begin{vmatrix} \bar{\underline{A}} & \bar{\underline{B}} & \bar{\underline{C}} \\ \bar{\underline{D}} & \bar{\underline{E}} & \bar{\underline{F}} \\ \bar{\underline{G}} & \bar{\underline{H}} & \bar{\underline{I}} \end{vmatrix} = 0$$

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the elements $\bar{A}$, $\bar{B}$, etc., of which are given explicitly on Pages 62 and 63 of AAF TR-4798. Therein effects of a geared tab are included. These effects are not considered here.

For convenience and computational purposes, some notation is employed here which is different from that used in AAF TR-4798. The present notation is adapted from some convenient and appropriate notation employed first by M. Yachter.

The equivalence of the two notations is stated below for those terms where they differ.

AAF TR-4798	THIS REPORT
M	$m(x)$
S_α	$S_\alpha(x)$
S_β	$S_\beta(x)$
$f_h(x)$	$f = f(x)$
$f_\alpha(x)$	$F = F(x)$
$f_\beta(x)$	1.0
$\int_0^b M[f_h(x)]^2 dx$	$\int_0^b m(x) f^2(x)\, dx$

$\int_0^L S_\alpha f_h(x) f_\alpha(x)\,dx$	$S_\alpha = \int_0^L S_\alpha(x) f(x) F(x)\,dx$
$\int_{L_1}^{L_2} S_\beta [f_h(x)][f_\beta(x)]\,dx$	$S_\beta = \int_{L_1}^{L_2} S_\beta(x) f(x)\,dx$
$\int_0^L I_\alpha [f_\alpha(x)]^2\,dx$	$I_\alpha = \int_0^L I_\alpha(x) F^2(x)\,dx$
$\int_{L_1}^{L_2} I_\beta [f_\beta(x)]^2\,dx$	$I_\beta = \int_{L_1}^{L_2} I_\beta(x)\,dx$
$\int_{L_1}^{L_2} [I_\beta + (c-a)b\,S_\beta] f_\alpha(x) f_\beta(x)\,dx$	$P_{\alpha\beta} = \int_{L_1}^{L_2} [S_\beta(x)(c-a)b + I_\beta(x)] F(x)\,dx$
$\left(\frac{\omega_h}{\omega_\alpha}\right)^2$	p
$\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2$	g
$\left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + jg) - \Omega$	Z

$L_h = K_1(L_h) + K_2(L_h)$	$1.0 + K_2(L_h)$
$L_\alpha = K_1(L_\alpha) + K_2(L_\alpha) + K_3(L_\alpha)$	$\frac{1}{2} + K_2(L_\alpha) + K_3(L_\alpha)$
$M_\alpha = K_1(M_\alpha) + K_2(M_\alpha)$	$\frac{3}{8} + K_2(M_\alpha)$

ω_h	$\frac{2\pi f_h}{60}$	where f_h, f_α and f_β are uncoupled frequencies in cpm.
ω_α	$\frac{2\pi f_\alpha}{60}$	
ω_β	$\frac{2\pi f_\beta}{60}$	

LIMITS OF INTEGRATION IN THIS REPORT

$0 - L \longrightarrow$ ROOT TO WING TIP

$L_1 - L_2 \longrightarrow$ INBOARD END OF AILERON TO OUTBOARD END

g = non-dimensional damping coefficient for entire wing coupled motion.

The following aerodynamic parts of the various determinant elements are defined for use here:

Wing Terms

$$A_{hh} = \int_0^l b^2 f^2(x)\,dx + b_r K_2(L_h)\int_0^l b f^2(x)\,dx$$

$$A_{h\alpha} = -\int_0^l a b^3 f(x)F(x)\,dx + b_r K_2(L_\alpha)\int_0^l b^2 f(x)F(x)\,dx + b_r^2 K_3(L_\alpha)\int_0^l b f(x)F(x)\,dx - b_r K_2(L_h)\int_0^l (\tfrac{1}{2}+a) b^2 f(x)F(x)\,dx$$

$$A_{\alpha h} = -\int_0^l a b^3 f(x)F(x)\,dx - b_r K_2(L_h)\int_0^l (\tfrac{1}{2}+a) b^2 f(x)F(x)\,dx$$

$$A_{\alpha\alpha} = \int_0^l (\tfrac{1}{8}+a^2) b^4 F^2_{(x)}\,dx + b_r K_2(M_\alpha)\int_0^l b^3 F^2_{(x)}\,dx + b_r K_2(L_h)\int_0^l (\tfrac{1}{2}+a)^2 b^3 F^2_{(x)}\,dx - b_r K_2(L_\alpha)\int_0^l (\tfrac{1}{2}+a) b^3 F^2_{(x)}\,dx - b_r^2 K_3(L_\alpha)\int_0^l (\tfrac{1}{2}+a) b^2 F^2_{(x)}\,dx$$

Aileron Terms

$$A_{h\beta} = \int_{l_1}^{l_2} [L_\beta - (c-e)L_z]\, b^3 f(x)\,dx$$

$$A_{\alpha\beta} = \int_{l_1}^{l_2} [M_\beta - (\tfrac{1}{2}+a)L_\beta - (c-e)M_z + (c-e)(\tfrac{1}{2}+a)L_z]\, b^4 F(x)\,dx$$

$$A_{\beta h} = \int_{l_1}^{l_2} [T_h - (c-e)P_h]\, b^3 f(x)\,dx$$

$$A_{\beta\alpha} = \int_{l_1}^{l_2} [T_\alpha - (c-e)P_\alpha - (\tfrac{1}{2}+a)T_h + (\tfrac{1}{2}+a)(c-e)P_h]\, b^4 F(x)\,dx$$

$$A_{\beta\beta} = \int_{l_1}^{l_2} [T_\beta - (c-e)(P_\beta + T_z) + (c-e)^2 P_z]\, b^4\,dx$$

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With the definitions made above, the determinant elements of AAF TR-4798 can be expressed as follows (geared tab effects are neglected altogether):

$$\bar{\underline{A}} = (1 - pZ)M + \pi\rho A_{hh}$$
$$\bar{\underline{B}} = S_{\alpha} + \pi\rho A_{h\alpha}$$
$$\bar{\underline{C}} = S_{\beta} + \pi\rho A_{h\beta}$$
$$\bar{\underline{D}} = S_{\alpha} + \pi\rho A_{\alpha h}$$
$$\bar{\underline{E}} = (1 - Z)I_{\alpha} + \pi\rho A_{\alpha\alpha}$$
$$\bar{\underline{F}} = P_{\alpha\beta} + \pi\rho A_{\alpha\beta}$$
$$\bar{\underline{G}} = S_{\beta} + \pi\rho A_{\beta h}$$
$$\bar{\underline{H}} = P_{\alpha\beta} + \pi\rho A_{\beta\alpha}$$
$$\bar{\underline{I}} = (1 - gZ)I_{\beta} + \pi\rho A_{\beta\beta}$$

Thus each term is divided into aerodynamic and mechanical parts.

Finally, the first row of determinant elements is divided through by M, the second by I_{α}, and the third by I_{β}. The resulting determinant has the form

$$\begin{vmatrix} \bar{A}_{hh} - pZ & \bar{A}_{h\alpha} & \bar{A}_{h\beta} \\ \bar{A}_{\alpha h} & \bar{A}_{\alpha\alpha} - Z & \bar{A}_{\alpha\beta} \\ \bar{A}_{\beta h} & \bar{A}_{\beta\alpha} & \bar{A}_{\beta\beta} - gZ \end{vmatrix} = 0$$

where the determinant elements are now the non-dimensional quantities:

$$\bar{A}_{hh} = 1 + \frac{\pi\rho A_{hh}}{M}$$
$$\bar{A}_{h\alpha} = \frac{S_{\alpha}}{M} + \frac{\pi\rho A_{h\alpha}}{M}$$
$$\bar{A}_{h\beta} = \frac{S_{\beta}}{M} + \frac{\pi\rho A_{h\beta}}{M}$$
$$\bar{A}_{\alpha h} = \frac{S_{\alpha}}{I_{\alpha}} + \frac{\pi\rho A_{\alpha h}}{I_{\alpha}}$$

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$$\bar{A}_{\alpha\alpha} = 1 + \frac{\pi\rho A_{\alpha\alpha}}{I_\alpha}$$

$$\bar{A}_{\alpha\beta} = \frac{P_{\alpha\beta}}{I_\alpha} + \frac{\pi\rho A_{\alpha\beta}}{I_\alpha}$$

$$\bar{A}_{\beta h} = \frac{S_\beta}{I_\beta} + \frac{\pi\rho A_{\beta h}}{I_\beta}$$

$$\bar{A}_{\beta\alpha} = \frac{P_{\alpha\beta}}{I_\beta} + \frac{\pi\rho A_{\beta\alpha}}{I_\beta}$$

$$\bar{A}_{\beta\beta} = 1 + \frac{\pi\rho A_{\beta\beta}}{I_\beta}$$

The expansion of the above determinant would ordinarily result in a cubic polynomial in Z. This can be solved by standard means. However, since the plynomial would have complex coefficients and the roots Z would be complex, it is considered simpler here to make assumptions which result in a quadratic in Z which is more easily solved.

The assumption consists in evaluating the term $\bar{A}_{\beta\beta} - gZ$ by assuming on the basis of experience that g = .05 in this term and the ratio $\frac{\omega_\beta}{\omega}$ has a definite value, where $\omega_\beta = \frac{2\pi f_\beta}{60}$; $\omega = \frac{2\pi f_c}{60}$

and f_c is the critical frequency of flutter oscillations in cpm. By assuming different frequency ratios $\frac{\omega_\beta}{\omega}$ SAY $\frac{\omega_\beta}{\omega} = 0, \frac{1}{2}, 1, 1.5$

a range of cases can be covered to include must cases encountered in the actual airplane. The term $A_{\beta\beta} - gZ$ is replaced by

$$\bar{A}'_{\beta\beta} = -\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 (1+.05j)\left(\frac{\omega_\alpha}{\omega}\right)^2 + \bar{A}_{\beta\beta} = \bar{A}_{\beta\beta} - \left(\frac{\omega_\beta}{\omega}\right)^2 (1+.05j)$$

which has a definite numerical value for each assumed frequency ratio $\frac{\omega_\beta}{\omega}$. The determinant then has the appearance

$$\begin{vmatrix} \bar{A}_{hh} - pZ & \bar{A}_{h\alpha} & \bar{A}_{h\beta} \\ \bar{A}_{\alpha h} & \bar{A}_{\alpha\alpha} - Z & \bar{A}_{\alpha\beta} \\ \bar{A}_{\beta h} & \bar{A}_{\beta\alpha} & \bar{A}'_{\beta\beta} \end{vmatrix} = 0$$

This is expanded to give the quadratic $Z^2 - 2\lambda Z + \eta = 0$

where

$$\lambda = -\frac{1}{2}\left\{\left[\bar{A}_{\alpha\beta}\bar{A}_{\beta\alpha} + \frac{1}{p}\bar{A}_{\beta h}\bar{A}_{h\beta}\right]\frac{1}{\bar{A}_{\beta\beta}} - \left[\bar{A}_{\alpha\alpha} + \frac{\bar{A}_{hh}}{p}\right]\right\}$$

$$\eta = \left\{\frac{1}{p\bar{A}_{\beta\beta}}\left[\bar{A}_{\alpha h}\bar{A}_{h\beta}\bar{A}_{\beta\alpha} + \bar{A}_{\beta h}\bar{A}_{h\alpha}\bar{A}_{\alpha\beta} - \bar{A}_{hh}\bar{A}_{\alpha\beta}\bar{A}_{\beta\alpha} - \bar{A}_{\beta h}\bar{A}_{h\beta}\bar{A}_{\alpha\alpha}\right] + \frac{1}{p}\left[\bar{A}_{hh}\bar{A}_{\alpha\alpha} - \bar{A}_{\alpha h}\bar{A}_{h\alpha}\right]\right\}$$

The solutions are $Z_1 = \lambda + \sqrt{\lambda^2 - \eta}$ AND $Z_2 = \lambda - \sqrt{\lambda^2 - \eta}$

Let

$$Z_1 = X_1 + jY_1 = \left(\frac{\omega_\alpha}{\omega_1}\right)^2 + jg_1\left(\frac{\omega_\alpha}{\omega_1}\right)^2$$

$$Z_2 = X_2 + jY_2 = \left(\frac{\omega_\alpha}{\omega_2}\right)^2 + jg_2\left(\frac{\omega_\alpha}{\omega_2}\right)^2$$

Then $g_1 = \frac{Y_1}{X_1}$; $g_2 = \frac{Y_2}{X_2}$

where g_1 and g_2 are the damping coefficients required for flutter to exist at a given $1/k$ value (one corresponds to each root Z).

Now

$$\frac{1}{k} \times \frac{1}{\sqrt{X_1}} = \frac{V_1}{b_r\omega_\alpha}$$

$$\frac{1}{k} \times \frac{1}{\sqrt{X_2}} = \frac{V_2}{b_r\omega_\alpha}$$

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where v_1 and v_2 are the flutter velocities associated with the roots Z_1, Z_2, for a given $1/k$ value.

For convenience in plotting final results, g (percent) versus $\frac{V}{f_\alpha}\frac{(mph)}{(cpm)}$ is desired for each $1/k$ value: g % = g x 100 %

$$\frac{V}{f_\alpha}\frac{(mph)}{(cpm)} \quad \frac{V\ (ft/sec)}{b_r\ (ft)\ \omega_\alpha\ (rad/sec)} \quad x \quad \frac{b_r\ (ft)}{11}$$

Bending Torsion Flutter

In case the aileron frequency can be assumed infinite (aileron rigid), the following simplification results; all determinant elements with a β subscript are zero. The determinant becomes, then

$$\begin{vmatrix} \bar{A}_{hh} - PZ & \bar{A}_{h\alpha} \\ \bar{A}_{\alpha h} & \bar{A}_{\alpha\alpha} - Z \end{vmatrix} = 0$$

which, on expansion, becomes $Z^2 - 2\lambda Z + \eta = 0$

where

$$\lambda = \frac{1}{2}\left[\bar{A}_{\alpha\alpha} + \frac{\bar{A}_{hh}}{P}\right]$$

$$\eta = \frac{1}{P}\left[\bar{A}_{hh}\ \bar{A}_{\alpha\alpha} - \bar{A}_{h\alpha}\ \bar{A}_{\alpha h}\right]$$

The solution proceeds as in the three-degree case.

Square Root of a Complex Number

The term $\sqrt{\lambda^2 - \eta}$ appears in the theory above, where $(\lambda^2 - \eta)$ in a complex number $R + j\ I$. The positive square root is taken as follows:

Let $\sqrt{R + j\ I} = R_o + jI_o$

Now $R + j\ I = \rho(\cos\theta + j\sin\theta) = \rho e^{j\theta}$

And $R_o + j\ I_o = \rho^{\frac{1}{2}}(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}) = \rho^{\frac{1}{2}} e^{j\frac{\theta}{2}}$

Now $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$; $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

Hence

$$R_o + j\, I_o = \rho^{\frac{1}{2}} \left(\pm\sqrt{\frac{1+\cos\theta}{2}} \pm j\sqrt{\frac{1-\cos\theta}{2}} \right)$$

$$= \rho^{\frac{1}{2}} \left(\pm\sqrt{\frac{1+R/\rho}{2}} \pm j\sqrt{\frac{1-R/\rho}{2}} \right) = \pm\sqrt{\frac{\rho+R}{2}} \pm j\sqrt{\frac{\rho-R}{2}}$$

But $\rho = \sqrt{R^2+I^2}$ then $R_o + j\, I_o = \pm\sqrt{\frac{\sqrt{R^2+I^2}+R}{2}} \pm j\sqrt{\frac{\sqrt{R^2+I^2}-R}{2}}$

Now if the "positive" root only is sought it is sufficient to confine the ambiguity of sign to a single term, say the first; thus

$$R_o = \pm\sqrt{\frac{\sqrt{R^2+I^2}+R}{2}}$$

$$I_o = +\sqrt{\frac{\sqrt{R^2+I^2}-R}{2}}$$

Further, it is easily seen that

$$R_o = \frac{\pm I}{2I_o} \quad (\text{for } I_o \neq 0)$$

To remove the ambiguity of sign in R_o, consider all possibilities:

R	I	R_o	I_o
>0	>0	>0	>0
<0	>0	>0	>0
<0	<0	<0	>0
>0	<0	<0	>0

It will be seen that I_o is always > o and the sign of R_o is always the same as the sign of I. Hence the following rule is set up:

1. Given $R + j\, I$; to find $\sqrt{R+jI} = (R_o + jI_o)$
2. Find $I_o = +\sqrt{\frac{\sqrt{R^2+I^2}-R}{2}}$
3. Find $R_o = \frac{I}{2I_o}$

2. Computation:

Before beginning computation, those wing modes to be considered should be selected. It is not possible to tell beforehand which modes are likely to result in the most critical flutter condition. Thus it is necessary to select various modes and perform separate analyses for each combination of modes chosen.

For the wing it is considered sufficient for a minimum analysis to make the following two selections of modes and an analysis made based on each combination:

(1) $f(x)$ first symmetric uncoupled bending mode

$F(x)$ first uncoupled torsion mode

(2) $f(x)$ first unsymmetric uncoupled bending mode

$F(x)$ first uncoupled torsion mode

The descriptions which follow pertain to the use of the tabular forms (Tables 1-7) in performing actual flutter computations.

Table 1

This table is designed to evaluate the wing integrals of both inertia and purely geometric content. Columns (11), (12), (13), (20), (21), (22), (23), (24), (25), (26), (27) are summed to obtain approximations of these integrals. Therein $f_s = f(x)$ is the uncoupled bending mode obtained from vibration analysis and $F_s = F(x)$ is the analogous uncoupled torsion mode. Throughout, the subscript s is similarly used to denote the value of the particular parameter designated at a given strip s.

Table 2

This table is designed to evaluate the aileron integrals $\int_{l_1}^{l_2}$ of both inertia and geometric content. Columns (4), (7), (13), (18), (19), (20) are summed to obtain these integrals. Columns (3), (4), (7), (11), (12), (13)

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of this table should be computed separately for each condition of aileron static balance.

Aileron static balance is specified in percent and is computed as follows: Let $(S_\beta)_0$ represent the unbalance (slug ft.) of the aileron without any balance weights. If a mass of balance weights of m_b slugs is placed x_b feet ahead of the aileron hinge, the resulting percent aileron balance is $\frac{m_b x_b}{(S_\beta)_0}$ x 100%. Often the aileron is stripwise balanced.

Table 3.

This table is designed to evaluate the wing aerodynamic terms.

$$A_{hh},\ A_{h\alpha},\ A_{\alpha h},\ A_{\alpha\alpha}$$

In setting up a tabular flutter analysis a useful series of values of the flutter parameter, $\frac{1}{k} = \frac{V}{b_r\omega}$ is chosen first. This runs across the top of Table 3 and subsequent tables. This parameter is a governing element in that it determines the evaluation of the flutter stability determinant at various points, one at a time. In the expression for $\frac{1}{k} = \frac{V}{b_r\omega}$, V is the velocity of the airplane (feet/second), b_r is the reference semi-chord, and ω is the flutter frequency (radians/sec). A suitable range for the 1/k parameter may be found for a given airplane as follows: Let v be chosen 50 to 100% higher than the maximum glide velocity of the airplane. Choose $\omega = \frac{2\pi f_\alpha}{60}$ where f_α is the uncoupled torsional frequency of the wing in cycles per minute as determined from vibration analysis or approximated from a vibration test. Calculate $\frac{V}{b_r\omega} = \frac{1}{k}$ using these values. Let this value be an upper limit to the values from the following set to fill in the top line of Table 3 and subsequent tables, being sure to include the calculated maximum 1/k at the right hand end: 0., 0.25, 0.5, 0.83, 1.25, 1.67, 2.00, 2.50, 2.94, 3.33, 3.75, 4.17, 5.00, 6.25, 8.33, 10.00.

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The left hand column of Table 3 identifies the items to be filled in in each line. The blanks in this column are filled in as follows:

Line	Item
①	$(b_r \times \Sigma$ ⑳, Table 1$) \times K_2(L_h)$
②	$A_{hh} =$ ① $+ (\Sigma$ ㉑, Table 1$)$
③	$(-b_r \times (\frac{1}{2}+a) \times \Sigma$ ㉖, Table 1$) \times K_2(L_h)$
④	$A_{\alpha h} =$ ③ $+ (-a \times \Sigma$ ㉗, Table 1$)$
⑤	$(b_r \times \Sigma$ ㉖, Table 1$) \times K_2(L_\alpha)$
⑥	$(b_r^2 \times \Sigma$ ㉕, Table 1$) \times K_3(L_\alpha)$
⑦	$A_{h\alpha} =$ ④ + ⑤ + ⑥
⑧	$(b_r \times \Sigma$ ㉓, Table 1$) \times K_2(M_\alpha)$
⑨	$(b_r \times (\frac{1}{2}+a) \times \Sigma$ ㉓, Table 1$) \times K_2(L_h)$
⑩	$(-b_r \times (\frac{1}{2}+a) \times \Sigma$ ㉓, Table 1$) \times K_2(L_\alpha)$
⑪	$(-b_r^2 \times (\frac{1}{2}+a)^2 \times \Sigma$ ㉒, Table 1$) \times K_3(L_\alpha)$
⑫	⑧ + ⑨ + ⑩ + ⑪ $+ (\frac{1}{8}+a^2) \times \Sigma$ ㉔, Table 1

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The items $K_2(L_h)$, $K_2(L_\alpha)$, $K_3(L_\alpha)$, $K_2(M_\alpha)$, are aerodynamic coefficients for tapered airfoils. These tables appear in this report as appendix VI, and are taken from AAF Technical Report 4798.

It will be noted that these values are complex numbers of the form $R + jI$ where R and I are ordinary real numbers and $j = \sqrt{-1}$.

Hence, in Table 3, appropriate entry blanks are provided, that for real numbers R being marked R and that for imaginary numbers (i.e., the number I multiplied by j) marked I.

Multiplication of a complex number $R + jI$ by a real number N is accomplished by multiplying R by N and I by N separately.

Addition of a real number N to a complex number $R + jI$ is accomplished by adding R and N algebraically and adding nothing to I.

Addition of two complex numbers $R_1 + jI_1$ and $R_2 + jI_2$ is accomplished by adding respectively real and imaginary parts, the result being $(R_1 + R_2) + j \quad (I_1 + I_2)$.

(In filling the table it is always convenient to carry along the columns $1/k = o$ for use later in calculation of coupled modes at zero airspeed.)

Table 4

This table is designed to calculate the aileron aerodynamic terms

$$A_{h\beta}\ ,\ A_{\alpha\beta}\ ,\ A_{\beta h},\ A_{\beta\alpha}\ ,\ A_{\beta\beta}$$

The left-hand column of Table 4 identifies the items to be filled in in each line. The blanks in this column are filled in as follows:

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Line	Item
(1)	$(\Sigma$ (18), Table 2$) \times L_\beta$
(2)	$(-(c-e) \times \Sigma$ (18), Table 2$) \times L_z$
(3)	$A_{h\beta} =$ (1) + (2)
(4)	$(\Sigma$ (20), Table 2$) \times M_\beta$
(5)	$(-(\frac{1}{2}+a) \times \Sigma$ (20), Table 2$) \times L_\beta$
(6)	$(-(c-e) \times \Sigma$ (20), Table 2$) \times M_z$
(7)	$((c-e)(\frac{1}{2}+a) \times \Sigma$ (20), Table 2$) \times L_z$
(8)	$A_{\alpha\beta} =$ (4) + (5) + (6) + (7)
(9)	$(\Sigma$ (18), Table 2$) \times T_h$
(10)	$(-(c-e) \times \Sigma$ (18), Table 2$) \times P_h$
(11)	$A_{\beta h} =$ (9) + (10)
(12)	$(\Sigma$ (20), Table 2$) \times T_\alpha$
(13)	$(-(c-e) \times \Sigma$ (20), Table 2$) \times P_\alpha$
(14)	$(-(\frac{1}{2}+a) \times \Sigma$ (20), Table 2$) \times T_h$
(15)	$((\frac{1}{2}+a)(c-e) \times \Sigma$ (20), Table 2$) \times P_h$
(16)	$A_{\beta\alpha} =$ (12) + (13) + (14) + (15)
(17)	$(\Sigma$ (19), Table 2$) \times T_\beta$
(18)	$(-(c-e) \times \Sigma$ (19), Table 2$) \times P_\beta$
(19)	$(-(c-e) \times \Sigma$ (19), Table 2$) \times T_z$
(20)	$((c-e)^2 \times \Sigma$ (19), Table 2$) \times P_z$
(21)	$A_{\beta\beta} =$ (17) + (18) + (19) + (20)

The items L_β, L_z, M_β, M_z, T_h, P_h, T_α, P_α, T_β, P_β, T_z, P_z are complex quantities. These are found, for various values of $1/k$, in the tables in Appendix VI. It is necessary to calculate the proper e value before use of these tables since the tables are prepared for various e values. (See Section A,-3."Notation") That tabular value of e may be used which is nearest to the calculated value. The same rules of operation apply to these

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complex quantities as described for Table 3.

Table 5

Wing Determinant Elements

This part of the table is designed to evaluate the wing determinant elements $\bar{A}_{hh}$, $\bar{A}_{h\alpha}$, $\bar{A}_{\alpha h}$, $\bar{A}_{\alpha\alpha}$ from the corresponding airforce terms. The left-hand column of Table 5 identifies items to be filled in in each line. The blanks in this column are filled in as follows:

Notation		Item
$\bar{A}_{hh}$	=	$1.00000 + \left(\frac{\pi\rho}{M}\right) \times A_{hh}$
$\bar{A}_{h\alpha}$	=	$\left(\frac{S_\alpha}{M}\right) + \left(\frac{\pi\rho}{M}\right) \times A_{h\alpha}$
$\bar{A}_{\alpha h}$	=	$\left(\frac{S_\alpha}{I_\alpha}\right) + \left(\frac{\pi\rho}{I_\alpha}\right) \times A_{\alpha h}$
$\bar{A}_{\alpha\alpha}$	=	$1.00000 + \left(\frac{\pi\rho}{I_\alpha}\right) \times A_{\alpha\alpha}$

A_{hh} is found in line ②, $A_{h\alpha}$ in line ⑦, $A_{\alpha h}$ in line ④ and $A_{\alpha\alpha}$ in line ⑫ of Table 3.

Aileron Determinant Elements

This part of the table is designed to be evaluated once for each percent of aileron static balance employed. In the design stage of the airplane it is well to assume a range of three or four different percents of balance, calculating the table once for each percent. Thus a final set of flutter results covering a wide range can be obtained and used to predict desired balance weights on the aileron. The left-hand column of Table 5 identifies items to be filled in in each line. The blanks in this column are filled in as follows:

Notation		Item
$\bar{A}_{h\beta}$	=	$\left(\frac{S_\beta}{M}\right) + \left(\frac{\pi\rho}{M}\right) A_{h\beta}$
$\bar{A}_{\alpha\beta}$	=	$\left(\frac{P_{\alpha\beta}}{I_\alpha}\right) + \left(\frac{\pi\rho}{I_\alpha}\right) A_{\alpha\beta}$
$\bar{A}_{\beta h}$	=	$\left(\frac{S_\beta}{I_\beta}\right) + \left(\frac{\pi\rho}{I_\beta}\right) A_{\beta h}$
$\bar{A}_{\beta\alpha}$	=	$\left(\frac{P_{\alpha\beta}}{I_\beta}\right) + \left(\frac{\pi\rho}{I_\beta}\right) A_{\beta\alpha}$
$\bar{A}_{\beta\beta}$	=	$1.00000 + \left(\frac{\pi\rho}{I_\beta}\right) A_{\beta\beta}$

The items A_{hh}, $A_{\alpha\beta}$, $A_{\beta h}$, $A_{\beta\alpha}$ and $A_{\beta\beta}$ are found respectively in lines ③, ⑧, ⑪, ⑯, ㉑ of Table 4. In Table 5, the auxiliary items $\pi\rho$, M, S_α, I_α, S_β, I_β, $P_{\alpha\beta}$

are entered. These are obtained as follows:

$\pi\rho$ is calculated as 3.14159 times the value of air density (slugs/ft^3) at the altitude selected for flutter study. Usually this altitude can be taken as 75% of the service ceiling of the aircraft.

M	=	Σ ⑪ ,	Table 1	
S	=	Σ ⑬ ,	Table 1	
I	=	Σ ⑫ ,	Table 1	
S	=	Σ ⑦ ,	Table 2	vary with % balance of aileron
I	=	Σ ④ ,	Table 2	
P	=	Σ ⑬ ,	Table 2	

<u>Table 6</u>

This table is designed to perform the final evaluation of the flutter stability determinant, the elements of the determinant having been developed in Tables 1 to 5.

The value of p = square of ratio of uncoupled bending frequency to uncoupled torsion frequency is entered on an auxiliary item on this Table.

The items of the various lines of Table 6 are self-explanatory as to indicated operations to be performed.

The product of two complex numbers is encountered for the first time. This is performed according to the following scheme:

$$(R_1 + j\ I_1) \times (R_2 + j\ I_2) = (R_1 R_2 - I_1 I_2) + j\ (I_1 R_2 + R_1 I_2)$$

If the numbers occur in the tabular form, as in Table 6, the multipliers paired to get the real entry may be schematically illustrated by:

R	I
R_1	I_1
|	|
R_2	I_2

R_1 R_2 - I_1 I_2 = Real Entr,

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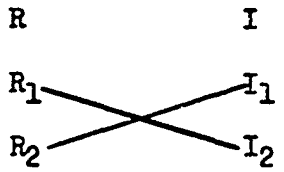
The multipliers paired to get the imaginary entry, likewise may be illustrated by:



Imaginary Entry ⟶ $R_1 \; I_2 + R_2 \; I_1$

The items of Table 6 are grouped so that unnecessary repetition of computation may be avoided as various cases of aileron balance and frequency ratio are employed in the flutter analysis. Lines (1) - (6) apply to the wing only and are therefore independent of aileron changes; lines (7) - (18) are independent of aileron frequency but vary with aileron static balance; lines (19) - (34) must be repeated for each variation of frequency ratio or static balance. Aid in the interpretation of the items of the various lines follows:

Line

(1) $\bar{A}_{hh}$ is obtained from Table 5 (top)

(2) (1) - $\bar{A}_{hh}$; algebraic subtraction, item by item

(3), (4) $\bar{A}_{hh} \times \bar{A}_{\beta\beta}$; complex multiplication; the necessary terms are found in Table 5 (top)

(5) (3) - (4) algebraic subtraction

(6) $\frac{1}{p} \times$ (5) ; real multiplication

(7), (8), (12), (11), commplex multiplication; the necessary terms appear in Table 5 (bottom)

(13), (16), (17), (14), (9), (18), (15), (10), self-explanatory: reel multiplication or addition

(19)

$$\bar{A}'_{\beta\beta} = \bar{A}_{\beta\beta} + \left(-\left[\frac{\omega_\beta}{\omega}\right]^2\right) + j\left(-.06 \times \left[\frac{\omega_\beta}{\omega}\right]^2\right)$$

The term in first parenthesis is added algebraically to the real entry, the second to the imaginary entry of $\bar{A}_{\beta\beta}$; $\left(\frac{\omega_\beta}{\omega}\right)$ is some previously assumed value.

(20) $\frac{1}{\bar{A}'_{\beta\beta}}$ the R and I items here refer to the entries in line 19

(21) Complex multiplication

(22), (23) Self-explanatory - real addition and multiplication

(24) Square of a complex number; complex multiplication of the number by itself; $(R+jI)^2 = (R^2 - I^2) + j(2RI)$

(25) Complex multiplication plus complex addition.

(26) Self-explanatory

(27) Square root of a complex number; see explanation for Table 7

(28) - (34) Self-explanatory

Table 7

This table is designed to develop by successive simple steps the square root of a complex number; this is of course also complex. The given number (from R and I entries of line (27), Table 6) is entered in columns (1) and (2), respectively. The other columns are explained by their headings. Columns (9) and (7) are the required R and I values of the square root of the given number.

Final Plot - Bending Torsion-Aileron Flutter

The final results are produced in the form of a graph having two curves on it. The higher curve is the critical one. The two curves are plotted from the respective pairs of values in lines (33) and (34), Table 6:

$$\left[\frac{V\,(\text{mph})}{\omega_\alpha\,(\text{cpm})}\ ;\ g\ \%\right]$$

The g values are taken as the ordinates, $\frac{V}{\omega_\alpha}$ being the abscissas.

Bending Torsion Flutter

In case a two-degree analysis (bending-torsion only with rigid aileron) is desired, Table 6 will provide the necessary elements. In this case, line (7) to (22), inclusive, are left blank. Line (23) is replaced by $-\frac{1}{2}$ x (2) = λ and line (25) is replaced by line (6). The computation then proceeds as before.

Coupled Modes

It is noted under the subject of vibration analysis that uncoupled bending and torsion modes are used in the flutter analysis. If $1/k = 0$ is used as one of the values of the flutter parameter, the tabular computation automatically provides the coupling of the uncoupled modes. The coupled frequencies at zero airspeed are given simply by

$$\frac{f_\alpha}{\sqrt{X_1}} \quad \text{and} \quad \frac{f_\alpha}{\sqrt{X_2}}$$

(see lines (31) and (32), Table 6)

The coupled modes associated with these coupled frequencies may be defined at any spanwise section x by the deflection of the elastic axis $h f(x)$ and the torsion about it $\alpha F(x)$ in the coupled mode. These results are obtained as follows, using values for $1/k = 0$.

Write the equations:

$$(\bar{A}_{hh} - PX)(h) + \bar{A}_{h\alpha}(\alpha) + \bar{A}_{h\beta}(\beta) = 0$$

$$\bar{A}_{\alpha h}(h) + (\bar{A}_{\alpha\alpha} - X)(\alpha) + \bar{A}_{\alpha\beta}(\beta) = 0$$

$$\bar{A}_{\beta h}(h) + \bar{A}_{\beta\alpha}(\alpha) + \bar{A}'_{\beta\beta}(\beta) = 0$$

(where $X = X_1$ or X_2 (lines (28) or (29), Table 6)

The $\bar{A}_{hh}$ terms, etc. are in Table 5 and $\bar{A}'_{\beta\beta}$ is line (19), Table 6, for $1/k = 0$.

For a given value of X, eliminate β from any two of the equations and solve for X in terms of h. Doing the same with any other pair of equations will merely provide a check on the same result. Say $\alpha = C_1 h$ is the result.

The modal pattern of the coupled mode of the wing may now be expressed in terms of the bending deflection of the elastic axis, which will have the deflection pattern $f(x)$ spanwise, plus a rotation about this axis of an amount. $C_1 F(x)$.

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Tables 1-7 are presented here in two forms, a set of blank tables and a set containing the parameters and actual complete calculation for a single configuration involved in flutter study on a given airplane. A g-v plot based on the results of lines (33) and (34) of Table 6 for this airplane is presented as Figure 6.

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Example of Tabular Flutter Analysis
Bending-Torsion-Aileron Flutter Stability Curves
50 % Aileron Static Balance - ALT. 10000 FT.

$\frac{\omega_\beta}{\omega} = \frac{1}{2}$

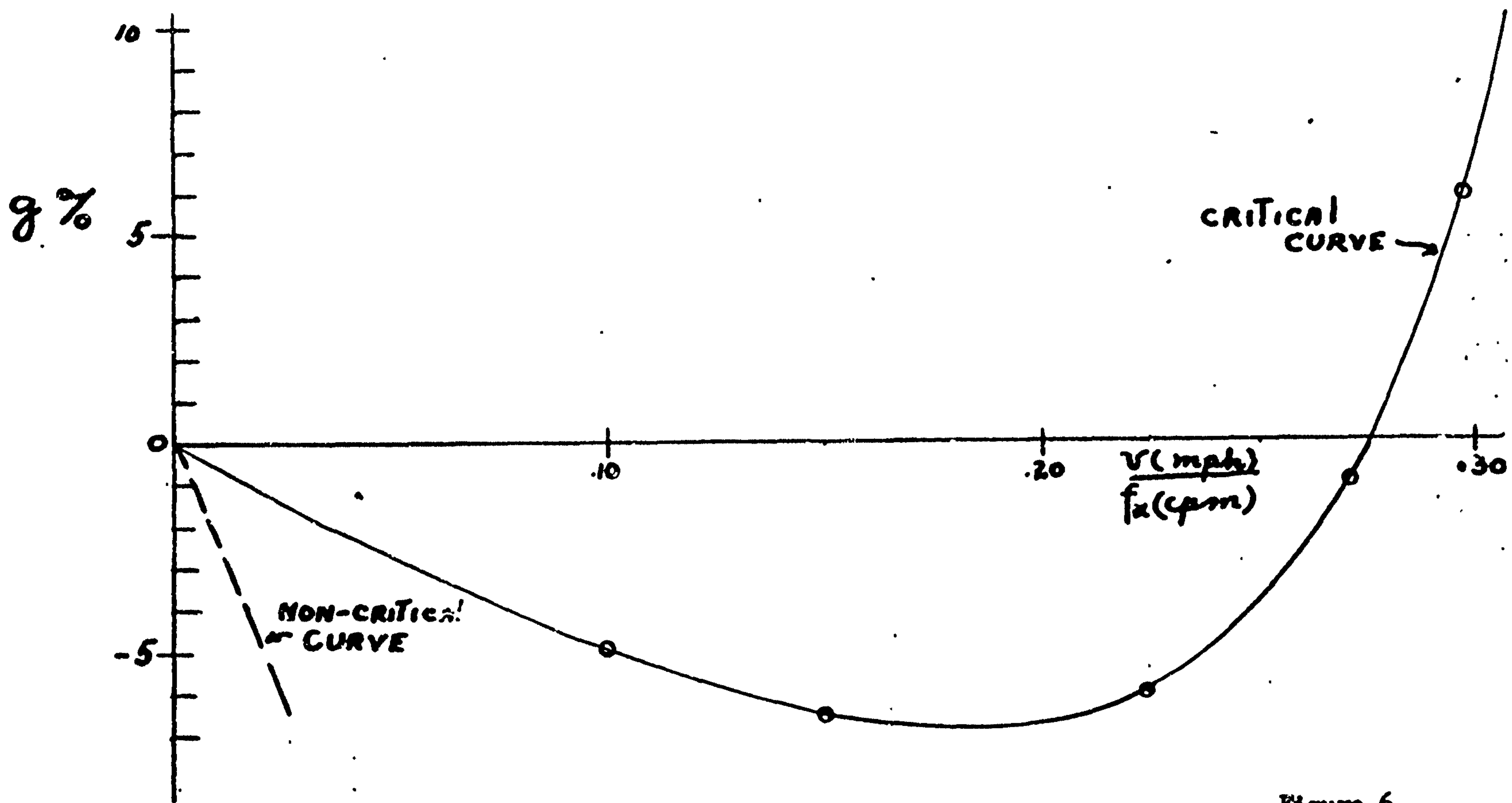

Figure 6

F. EMPENNAGE VIBRATION AND FLUTTER

The material developed and discussed in Sections A to E of this Report applies to the aircraft wing; however, with some changes in interpretation much of it can be applied to the empennage of the airplane. Hence, the purpose of this Section is to show briefly the means of applying the foregoing wing material to the tail surfaces. In this Section, completely detailed methods need therefore not be presented; instead, only those points unique to the empennage will be discussed and fitted into the previously developed analytical schemes. It is suggested that, in following this Report, analyses for the empennage be performed subsequent to wing analyses.

The various degrees of freedom and combinations thereof which can occur in the consideration of tail flutter are quite high in number. In this Section only three cases will be considered since it is felt that the cases chosen represent a minimum coverage of the problem for most airplanes. Unusual cases will always require special analyses, but these are not considered here.

Case 1. Fin Bending (h), fuselage side bending (α), rudder rotation(β)

In this case the fin-rudder is the aerodynamic surface involved in the flutter study. When the fin bends, its deflection corresponds to wing bending, the h degree of freedom; fuselage side bending changes the angle of attack of the fin; this corresponds to wing torsion, the α degree of freedom; the rudder plays the role of aileron, the β degree of freedom.

The (uncoupled) fin bending frequency and mode shape should first be determined by the same method as used to obtain the wing bending frequency. However, in this case the fin can be assumed cantilever from a rigid base, in which case no h_o terms need be carried ($h_o = 0$).

The(uncoupled) fuselage side bending frequency and mode (including the entire empennage as rigid) should next be determined by previous methods wherein the aft fuselage is first divided into (six) strips, etc. In this case the aft fuselage can be conveniently assumed cantilever from the trailing edge of the wing; fuselage forward of this arbitrary line can be neglected. Although this mode is a bending mode, it gives use to a torsion degree of freedom. Hence, all parameters must be interpreted in a torsional terms rather than translational. This can be done by plotting the fuselage side bending mode as illustrated schematically in Figure 7.

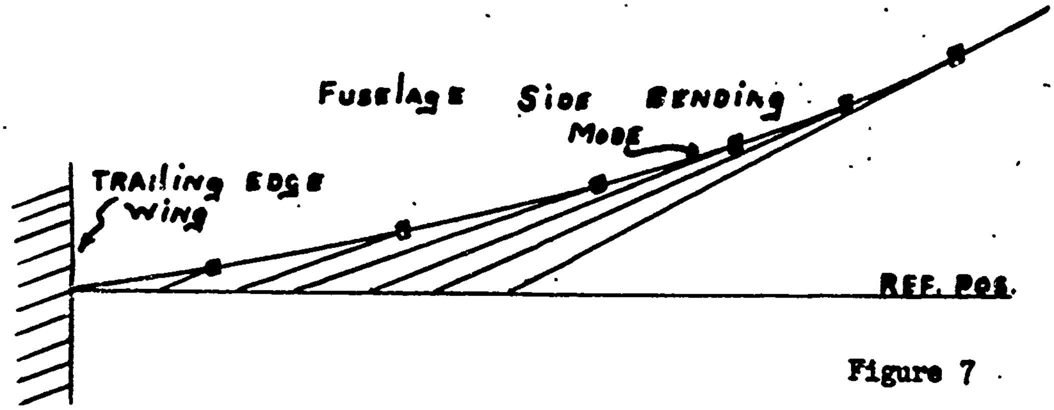

Figure 7

Draw a tangent, at the center of each fuselage strip, to the modal curve. Where this tangent intersects the reference position defines a rotation point for that strip. The chord line of the fin will, generally, lie along one such tangent and intersect the reference line at a point. This particular rotation point is the elastic axis of the aerodynamic surface. The moment of inertia I_α of the fuselage-empennage is

$$I_\alpha = \sum_s I_{\alpha s} F_s^2(x)$$

where $I_{\alpha s}$ is the moment of inertia of fuselage strip s about its particular rotation point in the fuselage bending mode; F_s ($=1$) for the aftmost (tail surface) fuselage strip; otherwise F_s (<1) is the ratio of that angular deflection pertaining to the strip s to that of the aftmost strip. The above calculation is based on the modal shape of the fuselage side bending mode (Figure 7).

The general geometric properties of the system are determined by the scheme of Figure 2, the midchord of the fin-rudder system being interpreted the same as that of wing-aileron, the above-defined rotation point being the elastic axis, etc. It will be noted that under these conditions the parameter a is almost invariably negative.

In the general flutter theory, the integrals of the "Wing Terms" are replaced by analogous integrals based on the fin, where $f(x)$ is the fin bending mode and $F(x) \equiv 1$. The reference semi-chord b_r is chosen as before. Similarly, the "Aileron Terms" are based on the rudder. Otherwise, everything proceeds analogously to the wing analysis and Tables 1 to 7 are applicable. In the event that the fin natural frequency is very high (say three times the fuselage side bending frequency) the fin may be considered rigid

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($f_h(x)=0$) with considerable resulting simplification. In this case the system becomes a two-degree-of-freedom one wherein the degrees of freedom are α and β.

<u>Case 2.</u> Stabilizer bending (h), fuselage vertical bending (α), elevator rotation (β).

In this case there is complete analogy with Case 1, wherein the whole stabilizer (both sides) plays the role of the fin, the elevators (both sides) play the role of the rudder, and fuselage vertical bending is strictly analogous to fuselage side bending.

<u>Case 3.</u> Fuselage torsion (h), fuselage side bending (α), rudder rotation (β).

For this calculation the natural frequency of the fuselage in uncoupled torsion is required. This can be calculated as in the case of wing torsion assuming again the fuselage cantilevered from the trailing edge of the wing.

In this case the fin-rudder is the aerodynamic surface involved in flutter. The aerodynamic effect of the stabilizer-elevator (which can be considered a single rigid unit) can also be included. The fin can now be considered rigid. The "bending" (h) mode f(x) of both fin and stabilizer-elevator is a straight line rotated about the fuselage torsional rotation point. The <u>aerodynamic</u> effect of the (rigid) stabilizer-elevator is included by adding to A_{hh} for the fin, a term A_{hh} based on the stabilizer-elevator and taking the sum of the two as the final A_{hh} for analysis. The mechanical effect of the "rigid" stabilizer-elevator is included by adding to $M = \int m(x) f(x)\, dx$ a similar term based on the entire horizontal tail (both sides); and to $S_\alpha = \int S_\alpha f(x)\; F(x)\; dx$ a term $\int S_\alpha f(x)\, dx$ for the entire stabilizer. The mode $f(x)$ (h deflection due to fuselage torsion) should be normalized on either fin or stabilizer (whichever has the greater value at its outboard strip). All other terms are based on the fuselage and fin rudder and are developed as described analogously for Cases 1 and 2.

APPENDIX I

FLUTTER TABLES - EXAMPLES

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WING FLUTTER CALCULATION
BENDING–TORSION–AILERON
WING INERTIA, DYNAMIC, & GEOMETRIC TERMS

TABLE I

STRIP	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬
	WING INTERVAL (INCHES)	MID-STA	MASS m_s	STATIC MOMENT $(S_\alpha)_s$	MASS MOMENT OF INERTIA $(I_\alpha)_s$	f_s	F_s	f_s^2	F_s^2	$f_s F_s$	$m_s f_s^2$	$(I_\alpha)_s F_s^2$	$(S_\alpha)_s f_s F_s$
S	STA-STA.		SLUGS	SLUG-FT	SLUG-FT²	BENDING MODE	TORSION MODE	⑥²	⑦²	⑥·⑦	③·⑧	⑤·⑨	④·⑩
0	0-22	℄ airplane	19 270	—	—	-.011	0	.000121	0	0	003332	0	0
1	22-43	32 88	1 659	1.082	1.543	-.190	-.149	036100	.022201	.028310	.059890	.030256	.029782
2	43-72	57 75	.390	.397	1.560	-.310	-.319	096100	101761	.098890	.037479	.158747	039259
3	72-101	86.50	.392	.399	1 570	-.322	-.217	.103684	.047089	.069874	.040644	.073930	.027880
4	101-131	116.00	.408	.415	1.625	+.046	-.032	002116	001024	-.001472	.000863	.001664	-.000611
5	131-161	146.00	.408	.415	1.625	+.598	+.460	.357604	.211600	.275080	145 902	.343850	114158
6	161-189	176 00	.503	.428	1.534	1.000	1.000	1 000000	1.000000	1.000000	.503000	1 534000	428000
											Σ⑪ = M	Σ⑫ = I_α	Σ⑬ = S_α
											= .79011	= 2.14645	= .63847

STRIP	⑭	⑮	⑯	⑰	⑱	⑲	⑳	㉑	㉒	㉓	㉔	㉕	㉖	㉗
	Δx_s	SEMI-CHORD b_s	$b_s \Delta x_s$	$b_s^2 \Delta x_s$	$b_s^3 \Delta x_s$	$b_s^4 \Delta x_s$	$f_s^2 b_s \Delta x_s$	$f_s^2 b_s^2 \Delta x_s$	$F_s^2 b_s^2 \Delta x_s$	$F_s^2 b_s^3 \Delta x_s$	$F_s^2 b_s^4 \Delta x_s$	$f_s F_s b_s \Delta x_s$	$f_s F_s b_s^2 \Delta x_s$	$f_s F_s b_s^3 \Delta x_s$
S	FEET	FEET	⑭·⑮	⑮·⑯	⑮·⑰	⑮·⑱	⑧·⑯	⑧·⑰	⑨·⑰	⑨·⑱	⑨·⑲	⑩·⑯	⑩·⑰	⑩·⑱
0	1 833	2.521	4.62099	11.64952	29.36844	74 03784	.000559	.001410	0	0	0	0	0	0
1	1.750	2.521	4.41175	11 12202	28.03861	70 68534	.159264	.401505	.246920	.622485	1.569285	.124897	.314864	.793773
2	2 417	2.521	6.09326	15 36111	38.72536	97 62663	.585562	1 476203	1.563162	3.940731	9 934582	.602562	1 519060	3 829551
3	2.417	2.521	6 09326	15.36111	38.72536	97 62663	.631774	1 592701	.723339	1 823538	4 597140	.425760	1 073342	2.705896
4	2.500	2.521	6.30250	15 88860	40 05516	100 97906	.013336	.033620	.016270	.041016	103403	-.009277	-.023388	-.058961
5	2.500	2.521	6.30250	15 88860	40.05516	100 97906	2.253799	5.681827	3 362028	8.475672	21 367169	1.733692	4 084641	11.018373
6	2.333	2 521	5 88149	14 82724	37.37947	94.23364	5 881490	14 827240	14 827240	37 379470	94 233640	5.881490	14 827240	37 379470
							Σ⑳ =	Σ㉑ =	Σ㉒ =	Σ㉓ =	Σ㉔ =	Σ㉕ =	Σ㉖ =	Σ㉗ =
				a = -.43801			9.52578	24 014500	20.73896	52.28291	131 80522	8.75912	21.79576	55.66810
				½+a = 06199										
				$(\frac{1}{2}+a)^2$ = .0038428										
				$\frac{1}{8}+a^2$ = .31685										

WING FLUTTER CALCULATION

BENDING-TORSION-AILERON

AILERON INERTIA, DYNAMIC, & GEOMETRIC TERMS (50)% STATIC BALANCE

TABLE 2

STRIP	(1) AILERON INTERVAL	(2) Δx_s	(3) STATIC UNBALANCE $(S_\beta)_s$	(4) MASS MOMENT OF INERTIA $(I_\beta)_s$	(5) WING BENDING MODE IN AILERON REGION	(6) WING TORSION MODE IN AILERON REGION	(7) $(S_\beta)_s f_s$	(8) WING SEMI-CHORD IN AILERON REGION-b_s	(9) $(c-a)_s$	(10) $(c-a)_s b_s$	(11) $(S_\beta)_s(c-a)_s b_s$	(12) $(S_\beta)_s(c-a)_s b_s + (I_\beta)_s$	(13) $[(S_\beta)_s(c-a)_s b_s + (I_\beta)_s] \cdot F_s$	(14) b_s^2	(15) b_s^3	(16) b_s^4
C	STA.-STA.	FEET	SLUG-FT.		f_s	F_s	(3) × (5)	FEET		FEET	(3) × (10)	(11) + (4)	(12) × (6)	$(8)^2$	(8) × (14)	(8) × (15)
1	101-116	1.25	.0081263	.0078287	-.103	-.106	-.0008370	2.5208	1.0479	2.6417	.0214672	.0292959	-.0031054	6.35443	16.01825	40.37878
2	116-131	"	.0081263	.0078287	.207	.043	+.0016821	"	"	"	.0214672	.0292969	+.0012597	"	"	"
3	131-146	"	.0081263	.0078287	.471	.277	.0038275	"	"	"	.0214672	.0292959	.0081150	"	"	"
4	146-161	"	.0035978	.0084892	.701	.660	.0025217	"	"	"	.0095030	.0179922	.0118748	"	"	"
5	161-176	"	-.0077252	.0101404	.908	.936	-.0070146	"	"	"	-.0204077	-.0102673	-.1096102	"	"	"
				$\Sigma(4) = I_r$			$\Sigma(7) = S_\beta$						$\Sigma(13) = P_{\alpha\beta}$			
				= .043116			= .0001798						= .0085339			

NOTE-- COLUMNS (3), (4), (7), (11), (12), (13) CHANGE WITH AILERON STATIC BALANCE; OTHERS DO NOT.

STRIP	(17) $b_s^3 f_s$	(18) $b_s^3 f_s \Delta x_s$	(19) $b_s^4 \Delta x_s$	(20) $b_s^4 F_s \Delta x_s$
S	(15) × (5)	(17) × (2)	(16) × (2)	(19) × (6)
1	-1.64988	-[illegible]06284	50.48950	-5.35189
2	+3.31878	4.14671	"	2.17105
3	7.54460	9.43298	"	13.98569
4	11.22879	14.03933	"	33.32307
5	14.54457	18.18503	"	47.25817
		$\Sigma(18) =$	$\Sigma(19) =$	$\Sigma(20) =$
		43.74021	252.4475	91.38599

WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
WING AERODYNAMIC TERMS

TABLE 8

	$1/k = \frac{v}{b\omega}$ →	(0) R	(0) I	(.5) R	(.5) I	(.833) R	(.833) I	(1.26) R	(1.26) I	(1.67) R	(1.67) I	(2.00) R	(2.00) I
①	(24.01450) · $K_h(L_h)$	0	0	-1.38564	-12.31704	-3.51020	-21.21273	-6.99422	-33.26609	-11.030.8	-46.33190	-14.47594	-57.43308
②	A_{hh} = ① + (24.01450)	24.01450	0	22.62886	-12.31704	20.50430	-21.21273	17.02028	-33.26609	12.98392	-46.33190	9.53856	-57.43308
③	(-3.45087) · $K_h(L_h)$	0	0	.19912	1.76995	.50441	3.04826	1.00507	4.78032	1.58509	6.65787	2.08018	8.25310
④	$A_{\alpha h}$ = ③ + (24.38318)	24.38318	0	24.58230	1.76995	24.88759	3.04826	25.38825	4.78032	25.96827	6.65787	26.46336	8.25310
⑤	(55.66810) · $K_h(L_\alpha)$	0	0	-2.31206	-56.38602	-8.18701	-95.56320	-16.21333	-146.69636	-25.57003	-200.18249	-33.55673	-244.47202
⑥	(55.66810) · $K_\alpha(L_\alpha)$	0	0	-14.27608	1.60602	-40.97729	6.77982	-96.39266	20.26653	-179.00412	43.61671	-266.27166	67.11846
⑦	$A_{h\alpha}$ = ④ + ⑤ + ⑥	24.38318	0	7.09417	-53.01025	-24.22671	-85.73512	-87.21774	-121.65251	-178.60588	-150.90791	-273.36503	-169.10547
⑧	(131.80832) · $K_\alpha(M_\alpha)$	0	0	0	-65.90261	0	-109.83724	0	-164.75653	0	-219.67581	0	-263.61044
⑨	(.50650) · $K_h(L_h)$	0	0	-.02923	-.25978	-.07404	-.44741	-.14752	-.70163	-.23265	-.97721	-.30532	-1.21135
⑩	(-8.17061) · $K_h(L_\alpha)$	0	0	.47144	8.27601	1.19430	14.02616	2.37969	21.53160	3.75301	29.38151	4.92524	35.88205
⑪	(-8.17061) · $K_\alpha(L_\alpha)$	0	0	2.09535	-.23572	6.01439	-.99510	14.14790	-2.97459	26.27309	-6.25801	39.08166	-9.85049
⑫	$A_{\alpha\alpha}$ = ⑧ + ⑨ + ⑩ + ⑪ + (41.76248)	41.76248	0	44.30004	-58.12710	48.89713	-97.25359	58.14235	-146.90115	71.55593	-197.52652	85.46406	-238.79023

WING FLUTTER CALCULATION

BENDING-TORSION-AILERON

AILERON AERODYNAMIC TERMS

TABLE 4

	$1/k = \frac{v}{b\omega}$ →	(0) R	(0) I	(.5) R	(.5) I	(.833) R	(.833) I	(1.25) R	(1.25) I	(1.67) R	(1.67) I	(2.00) R	(2.00) I	
(1)	(+3.74021)·L_β	1.75311	0	-5.60137	-8.19530	-19.17702	-11.86978	-47.00629	-13.51879	-88.02692	-11.30920	-151.11215	-6.61571	
(2)	(-3.1799)·L_z	-.62167	0	-.50993	.99327	-.33858	1.71063	-.08776	2.68263	.26778	3.73626	.54570	4.63146	
(3)	$A_{h\beta}$ = (1) + (2)	1.12144	0	-6.11130	-7.15203	-19.51655	-10.15915	-47.09404	-10.83616	-87.75914	-7.57102	-150.56648	-1.98425	←
(4)	(91.38599)·M_β	3.27802	0	-6.16947	-15.23039	-22.96164	-25.38387	-55.76556	-38.07597	-101.69885	-50.76857	-147.87441	-60.93156	
(5)	(-5.66502)·L_β	-.22705	0	+.72546	+1.05494	2.48371	1.53732	6.08803	1.75089	11.40470	1.46446	16.98101	.85683	
(6)	(-6.64376)·M_z	-1.10725	0	-1.10725	1.37360	-1.10725	2.28931	-1.10725	3.43403	-1.10726	4.57868	-1.10725	5.49439	
(7)	(.41185)·L_z	.08052	0	.06604	-.12865	.04385	-.22155	.00748	-.34744	-.03468	-.48391	-.07068	-.59985	
(8)	$A_{\alpha\beta}$ = (4)+(5)+(6)+(7)	2.02424	0	-6.48522	-12.93050	-21.54133	-21.77829	-50.79226	-33.23849	-91.42608	-45.20934	-132.07133	-55.17019	←
(9)	(42.74021)·T_h	1.75311	0	1.72472	-.25238	1.68116	-.43465	1.60981	-.68165	1.52710	-.94938	1.45651	-1.17683	
(10)	(-3.1799)·P_h	-.62167	0	-.61108	.09406	-.59486	.16198	-.56528	.25404	-.53742	.35380	-.51114	.43857	
(11)	$A_{\beta h}$ = (9) + (10)	1.13144	0	1.11364	-.15832	1.08629	-.27267	1.04153	-.42761	.98967	-.59558	.94537	-.73826	←
(12)	(91.38699)·T_α	3.27802	0	2.95497	-3.64676	2.37092	-6.03148	1.19834	-8.92275	-.50006	-11.69575	-2.25906	-13.81573	
(13)	(-6.64376)·P_α	-1.10725	0	-.98693	1.29268	-.76928	2.13730	-.33232	3.15951	.30070	4.13687	.95630	4.88863	
(14)	(-5.66502)·T_h	-.22705	0	-.22338	.03269	-.21774	.05629	-.20850	.08828	-.19778	.12296	-.18864	.15242	
(15)	(.41185)·P_h	.08052	0	.07915	-.01218	.07704	-.02098	.07360	-.03290	.06961	-.04582	.06620	-.05680	
(16)	$A_{\beta\alpha}$ = (12)+(13)+(14)+(15)	2.02424	0	1.82231	-2.33257	1.46094	-3.85887	.73112	-5.70786	-.32753	-7.47974	-1.42620	-8.83648	←
(17)	(252.4475)·T_β	1.36069	0	-.62127	-6.38315	-4.17750	-8.81143	-11.20892	-12.93869	-21.18893	-16.81528	-31.33882	-19.72624	
(18)	(-18.38293)·P_β	-.39110	0	.48415	1.73766	2.05241	2.85480	5.14708	4.17841	9.53049	5.40861	13.98108	6.32222	
(19)	(-18.35293)·T_z	-.3940	0	-.38385	.28326	-.37271	.47567	-.35449	.72109	-.33334	.97182	-.31580	1.17587	
(20)	(1.33426)·P_z	.12703	0	.12434	-.13543	.12019	-.21037	.11340	-.31839	.10581	-.42839	.09479	-.31763	
(21)	$A_{\beta\beta}$ = (17)+(18)+(19)+(20)	.70652	0	-.39663	-3.48765	-2.37761	-5.69133	-6.30293	-8.85758	-11.88627	-10.86324	-17.67426	-12.74578	←

$c - e = \frac{c_\beta - e_\beta}{b} = \frac{18.45'' - 16.25''}{30.25''} = .0727$

$e = .5372$; USE $e = .5$

I-4

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$\pi\rho$ = .006057 S_α = .63847
M = .79011 I_α = 2.14046

WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
WING DETERMINANT ELEMENTS

TABLE 5

$1/k = \frac{v}{b_r\omega}$ →	(0)		(.5)		(.833)		(1.25)		(1.67)		(2.00)	
	R	I	R	I	R	I	R	I	R	I	R	I
$\bar{A}_h$ = 1.0 + (.0076660)·A_{hh}	1.18910	0	1.17347	-.09442	1.18719	-.16262	1.13048	-.28502	1.09983	-.35510	1.07312	-.44020
$\bar{A}_{h\alpha}$ = (.80808)+(.0076660)$A_{h\alpha}$	.99499	0	.86246	-.40638	.62236	-.65725	.13947	-.93259	-.56111	-1.15686	-1.28754	-1.29636
$\bar{A}_{\alpha h}$ = (.39706)+(.0020219)·$A_{\alpha h}$	.36626	0	.36682	.00499	.36768	.00860	.36909	.01349	.37073	.01879	.37213	.02329
$\bar{A}_{\alpha\alpha}$ = 1.0 + (.0108219)·$A_{\alpha\alpha}$	1.11785	0	1.12501	-.16901	1.13798	-.27404	1.16407	-.41454	1.20192	-.55700	1.24117	-.67384

AILERON DETERMINANT ELEMENTS
(50)% STATIC BALANCE

$1/k = \frac{v}{b_r\omega}$ →	(0)		(.6)		(.833)		(1.25)		(1.67)		(2.00)	
	R	I	R	I	R	I	R	I	R	I	R	I
$\bar{A}_{h\beta}$ = (.0002'1756)+(.0076660)·$A_{h\beta}$	.00890	0	-.09662	-.05403	-.14938	-.07788	-.36056	-.08307	-.69276	-.05804	-1.00069	-.01521
$\bar{A}_{\alpha\beta}$ = (.0029758)+(.0020219)·$A_{\alpha\beta}$	.00969	0	-.01432	-.03649	-.05681	-.06146	-.13935	-.09380	-.25402	-.12758	-.36871	-.15548
$\bar{A}_{\beta h}$ = (.0002692)+(.14382)·$A_{\beta h}$	.16699	0	.16443	-.02277	.16050	-.03922	.15406	-.06150	.14660	-.08566	.14023	-.10618
$\bar{A}_{\beta\alpha}$ = (.20263)+(.14382)·$A_{\beta\alpha}$	.49376	0	.46493	-.33561	.41274	-.5545	.30778	-.82090	.15562	-1.07574	-.00434	-1.27006
$\bar{A}_{\beta\beta}$ = (1.0)+(.14382)·$A_{\beta\beta}$	1.10147	0	.94496	-.49728	.65805	-.81853	.09351	-1.20199	-.70998	-1.56235	-1.52753	-1.83510

S_β = .0001798
I_β = .042116
$P_{\alpha\beta}$ = .0005339

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WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
EVALUATION OF DETERMINANT

TABLE 6

$\mu = \left(\frac{\omega}{\omega_\alpha}\right)^2 = \left[\frac{670}{350}\right]^2 = .24631$

$1/\mu = 4.05992$

ITEMS INDEPENDENT OF AILERON STATIC BALANCE AND AILERON FREQUENCY

No.	$1/k = \frac{v}{b_r \omega}$	(0) R	(0) I	(.50) R	(.50) I	(.835) R	(.835) I	(1.25) R	(1.25) I	(1.67) R	(1.67) I	(2.00) R	(2.00) I
1	$-1/\mu \cdot \bar{A}_{hh} = (-4.05992) \cdot \bar{A}_{hh}$	-4.80735	0	-4.75419	+.38334	-4.61810	+.66022	-4.58966	+1.03536	-4.44400	+1.44200	-4.35678	+1.78750
2	① $-\bar{A}_{\alpha\alpha}$	-5.92520	0	-5.88920	.54736	-5.83608	.93466	-5.75373	1.44990	-5.66592	1.99940	-5.59795	2.46134
3	$\bar{A}_{h\alpha} \cdot \bar{A}_{\alpha h}$	1.32365	0	1.30468	-.29864	1.27333	-.50264	1.21024	-.76549	1.12357	-1.03978	1.02525	-1.26957
4	$\bar{A}_{\alpha h} \cdot \bar{A}_{h\alpha}$	.36443	0	.31840	-.14476	.23448	-.23631	.06406	-.34233	-.18428	-.43943	-.44894	-.51240
5	③ − ④	.95922	0	.98628	-.11392	1.03776	-.26628	1.14618	-.42316	1.30985	-.60035	1.48419	-.75717
6	$1/\mu \cdot$ ⑤ $= (4.05992) \cdot$ ⑤	3.89436	0	4.00422	-.46490	4.21310	-1.08128	4.65340	-1.71800	5.31789	-2.43737	6.02564	-3.07405

ITEMS INDEPENDENT OF AILERON FREQUENCY (50)% AILERON STATIC BALANCE

No.	$1/k = \frac{v}{b_r \omega}$	(0) R	(0) I	(.50) R	(.50) I	(.833) R	(.833) I	(1.25) R	(1.25) I	(1.67) R	(1.67) I	(2.00) R	(2.00) I
7	$\bar{A}_{h\beta} \cdot \bar{A}_{\beta h}$	.00478	0	-.01890	-.01316	-.05756	.06616	-.11989	.08582	-.17675	.25342	-.29698	.46594
8	$\bar{A}_{\beta h} \cdot \bar{A}_{h\beta}$	.00149	0	-.00891	-.00796	-.02703	-.00664	-.06066	+.00938	-.10360	.04912	-.14194	.10412
9	$1/\mu \cdot$ ⑧ $= (4.05992) \cdot$ ⑧	.00605	0	-.03617	-.03228	-.10974	-.02696	-.24627	.03808	-.42061	.19942	-.57627	.42272
10	⑦ + ⑨	.01083	0	-.05507	-.04404	-.16730	-.02080	-.36616	.12360	-.59736	.45234	-.77325	.89166
11	$\bar{A}_{\alpha h} \cdot \bar{A}_{h\beta}$	.00326	0	-.01683	-.02035	-.05425	-.03992	-.13196	-.03552	-.24832	-.02416	-.37203	-.02597
12	⑪ $\cdot \bar{A}_{\beta\alpha}$	.00161	0	-.01465	-.00381	-.03900	.01776	-.06977	.09739	-.07537	.26182	-.03595	.47207
13	$\bar{A}_{\beta h} \bar{A}_{h\alpha}$	.16615	0	.13256	-.08646	.07411	-.12990	-.03587	-.15225	-.18136	-.12153	-.31820	-.04508
14	⑬ $\cdot \bar{A}_{\alpha\beta}$	.00161	0	-.00505	-.00360	-.01219	.00282	-.00928	.02450	.03056	.05401	.11031	.06616
15	$-\bar{A}_{\alpha\alpha} \cdot$ ⑦	-.00666	0	+.02333	.01248	.06561	-.01649	.11372	-.12725	.16433	-.34142	.00492	-.68996
16	$-\bar{A}_{hh} \cdot$ ⑧	-.00167	0	.01132	.00748	.03268	.00014	.06672	-.03606	.09714	-.11678	.10601	-.22488
17	⑫ + ⑭ + ⑮ + ⑯	-.00411	0	.01196	.01255	.04700	.00423	.10139	-.04134	.15666	-.14237	.18529	-.27581
18	$1/\mu \cdot$ ⑰ $= (4.05992) \cdot$ ⑰	-.01669	0	.06074	.05096	.19082	.01717	.41164	-.16784	.63603	-.57801	.78226	-1.11977

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WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
EVALUATION OF DETERMINANT

TABLE 6 (CONT'D)

ITEMS DEPENDENT ON AILERON FREQUENCY AND STATIC BALANCE: $(\omega_\beta/\omega_\alpha) = (1/2)$; (50)% STATIC BALANCE

No.	$1/k = \frac{v}{b\omega}$ →	(0) R	(0) I	(.50) R	(.50) I	(.833) R	(.833) I	(1.25) R	(1.25) I	(1.67) R	(1.67) I	(2.30) R	(2.30) I
19	$\bar A_{pp}$ = $\bar A_{pp}$ + (-.27000)(k)(-.01887) = R + iI	.85157	0 †	64.196	-.50978	.40905	-.83103	-.15649	-1.2449	-.95908	-1.57485	-1.77753	-1.84560
20	Ω/ω_m = ...	1.17444	0	.45634	.58823	.07607	.96956	-.10496	.59994	-.28214	.46309	-.27072	.27109
21	(20) · (16)	.01277	0	-.02045	-.07964	-.05948	-.17211	-.06190	-.30947	-.04117	-.40440	-.04130	-.45874
22	(21) + (2)	-5.91248	0	-5.91915	.46781	-5.89556	.76255	-5.81563	1.14043	-5.70709	1.59500	-5.63925	2.00260
23	$-\frac{1}{2}$ (22) = λ	2.95624	0	2.95508	-.23391	2.94778	-.38128	2.90782	-.57022	2.85355	-.79750	2.81963	-1.00130
24	λ^2 = (23)^2	8.73945	0	8.67718	-1.38243	8.54403	-2.24786	8.13027	-3.31619	7.50674	-4.55140	6.94771	-5.64650
25	η = (20) · (16) + (6)	3.87476	0	4.03690	-.53535	4.28738	-.88809	4.74638	-1.38460	5.40611	-1.99975	6.13679	-2.55945
26	(24) − (25) = $\lambda^2 - \eta$	4.86459	0	4.65170	-.84709	4.25665	-1.35977	3.38389	-1.93159	2.10063	-2.57165	.81092	-3.08713
27	$\sqrt{\lambda^2 - \eta} = \sqrt{(26)}$ *	2.20558	0	-2.16498	.19558	-2.08661	.32582	-1.90789	.50621	-1.64638	.78100	-1.41470	1.09109
28	$\lambda + \sqrt{\lambda^2 - \eta}$ = (23) + (27) = $X_1 + iY_1$	5.16182	0	.79550	-.03923	.85917	-.05576	.99993	-.06401	1.20717	-.01650	1.40493	.08979
29	$\lambda - \sqrt{\lambda^2 - \eta}$ = (23) − (27) = $X_2 + iY_2$	.75066	0	5.12066	-.42949	5.03439	-.70680	4.81571	-1.07663	4.49993	-1.57850	4.23433	-2.09239
30	$(\omega_h/\omega_\alpha)^2 \, 1/k^2$ = (.8006) · $1/k^2$	0		.99003		.14999		.22500		.30070		.36012	
31	$\sqrt{X_1} = \sqrt{\text{Real part of (28)}}$	2.27196		.88854		.92697		.99996		1.09871		1.10510	
32	$\sqrt{X_2} = \sqrt{\text{Real part of (29)}}$	.86641		2.26289		2.24419		2.19447		2.12130		2.05775	
33	(30) ÷ (31) = $\frac{Y_1}{X_1}$; [I (28)] + [R (28)] = g	0	0	.10133	-.04855	.16182	-.06410	.22609	-.06401	.27368	-.01367	.30382	.06201
34	(30) ÷ (32) = $\frac{Y_2}{X_2}$; [I (29)] + [R (29)] = g	0	0	.03979	-.08387	.06683	-.14034	.10257	-.22352	.14175	-.35078	.17501	-.49416

* SEE TABLE 7

† For $1/k = 0$, $\bar A_{pp} = \bar A_{pp}$ (...)

COUPLED FREQUENCIES AT ZERO AIRSPEED:

$f_1 = \frac{1350}{2.27196} = 594$ cpm

$f_2 = \frac{1350}{.86641} = 1558$ cpm

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AUXILIARY WORK SHEET (TABLE 7)

SQUARE ROOT OF THE COMPLEX NUMBER R + jI

	①	②	③	④	⑤	⑥
	GIVEN	NUMBER				
1/λ ↓	R	I	R^2+I^2	$\sqrt{③}$	④-①	½·⑤
0	4.06059	0	—	—	—	—
.50	4.65178	-.84709	22.35162	4.72828	.07650	.03825
.833	4.25665	-1.35977	19.96904	4.46856	.21191	.10596
1.25	3.30389	-1.95159	15.28175	3.81638	.51249	.25625
1.67	2.10063	-2.57166	11.02603	3.32055	1.21992	.60996
2.00	.81092	-2.08713	10.18796	3.19186	2.38094	1.19047

	⑧	⑨	⑦
1/λ ↓	2.00·⑦	② ÷ ⑧ SQUARE	$\sqrt{⑥}$ ROOT
	—	R_0	I_0
0	—	2.20658	—
.50	.39116	-2.16558	.19558
.833	.65104	-2.08863	.32552
1.25	1.01242	-1.92789	.50621
1.67	1.56200	-1.64638	.78100
2.00	2.18218	-1.41470	1.09109

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WING FLUTTER CALCULATION

BENDING–TORSION–AILERON

WING INERTIA, DYNAMIC, & GEOMETRIC TERMS

TABLE I

STRIP	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)			
	WING INTERVAL	MID-STA	MASS m_s	STATIC MOMENT $(S_\alpha)_s$	MASS MOMENT OF INERTIA $(I_\alpha)_s$	f_s	F_s	f_s^2	F_s^2	$f_s F_s$	$m_s f_s^2$	$(I_\alpha)_s F_s^2$	$(S_\alpha)_s f_s F_s$			
s	STA.-STA.		SLUGS	SLUG-FT	SLUG-FT²	BENDING MODE	TORSION MODE	(6)²	(7)²	(6)×(7)	(3)×(8)	(5)×(9)	(4)×(10)			
0																
1																
2																
3																
4																
5																
6																
											Σ(11) = M	Σ(12) = I_α	Σ(13) = S_α			

STRIP	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)		
	Δx_s	SEMI-CHORD b_s	$b_s \Delta x_s$	$b_s^2 \Delta x_s$	$b_s^3 \Delta x_s$	$b_s^4 \Delta x_s$	$f_s^2 b_s \Delta x_s$	$f_s^2 b_s^2 \Delta x_s$	$F_s^2 b_s^2 \Delta x_s$	$F_s^2 b_s^3 \Delta x_s$	$F_s^2 b_s^4 \Delta x_s$	$f_s F_s b_s \Delta x_s$	$f_s F_s b_s^2 \Delta x_s$	$f_s F_s b_s^3 \Delta x_s$		
s	FEET	FEET	(14)×(15)	(15)×(16)	(15)×(17)	(15)×(18)	(8)×(16)	(8)×(17)	(9)×(17)	(9)×(18)	(9)×(19)	(10)×(16)	(10)×(17)	(10)×(18)		
0																
1																
2																
3																
4																
5																
6																
							Σ(20) =	Σ(21) =	Σ(22) =	Σ(23) =	Σ(24) =	Σ(25) =	Σ(26) =	Σ(27) =		
				a =												
				$\frac{1}{2}+a$ =												
				$(\frac{1}{2}+a)^2$ =												
				$\frac{1}{8}+a^2$ =												

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WING FLUTTER CALCULATION

BENDING-TORSION-AILERON

AILERON INERTIA, DYNAMIC, & GEOMETRIC TERMS ( )% STATIC BALANCE

TABLE 2

STRIP	① AILERON INTERVAL	② Δx_s	③ STATIC UNBALANCE $(S_\beta)_s$	④ MASS MOMENT OF INERTIA $(I_\beta)_s$	⑤ WING BENDING MODE IN AILERON REGION	⑥ WING TORSION MODE IN AILERON REGION	⑦ $(S_\beta)_s f_s$	⑧ WING SEMI-CHORD IN AILERON REGION-b_s	⑨ $(c-a)_s$	⑩ $(c-a)_s b_s$	⑪ $(S_\beta)_s (c-a)_s b_s$	⑫ $(S_\beta)_s (c-a)_s b_s + (I_\beta)_s$	⑬ $[(S_\beta)_s (c-a)_s b_s + (I_\beta)_s] F_s$	⑭ b_s^2	⑮ b_s^3	⑯ b_s^4
S	STA-STA	FEET	SLUG-FT.		f_s	F_s	③ × ⑤	FEET			③ × ⑩	⑪ + ④	⑫ × ⑥	⑧²	⑧ × ⑭	⑧ × ⑮
1																
2																
3																
4																
5																
				Σ④ = I_β			Σ⑦ = S_β						Σ⑬ = $P_{\alpha\beta}$			

NOTE — COLUMNS ③, ④, ⑦, ⑪, ⑫, ⑬ CHANGE WITH AILERON STATIC BALANCE; OTHERS DO NOT.

STRIP	⑰ $b_s^3 f_s$	⑱ $b_s^3 f_s \Delta x_s$	⑲ $b_s^4 \Delta x_s$	⑳ $b_s^4 F_s \Delta x_s$
S	⑮ × ⑤	⑰ × ②	⑯ × ②	⑲ × ⑥
1				
2				
3				
4				
5				
		Σ⑱ =	Σ⑲ =	Σ⑳ =

WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
WING AERODYNAMIC TERMS

TABLE 3

	$\frac{1}{k} = \frac{v}{b\omega}$ →	( )		( )		( )		( )		( )		( )		
		R	I	R	I	R	I	R	I	R	I	R	I	
①	( ) $\cdot K_0(L_h)$													
②	A_{hh} = ① + ( )													
③	( ) $\cdot K_0(L_h)$													
④	$A_{\alpha h}$ = ③ + ( )													
⑤	( ) $\cdot K_0(L_\alpha)$													
⑥	( ) $\cdot K_0(L_\alpha)$													
⑦	$A_{h\alpha}$ = ④ + ⑤ + ⑥													
⑧	( ) $\cdot K_0(M_\alpha)$													
⑨	( ) $\cdot K_0(L_h)$													
⑩	( ) $\cdot K_0(L_\alpha)$													
⑪	( ) $\cdot K_0(L_\alpha)$													
⑫	$A_{\alpha\alpha}$ = ⑧ + ⑨ + ⑩ + ⑪ + ( )													

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WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
AILERON AERODYNAMIC TERMS

TABLE 4

	$1/k = \frac{v}{b_r \omega}$ →	(	)	(	)	(	)	(	)	(	)	(	)	
		R	I	R	I	R	I	R	I	R	I	R	I	
①	( )·L_β													
②	( )·L_z													
③	$A_{h\beta}$ = ① + ②													
④	( )·M_β													
⑤	( )·L_β													
⑥	( )·M_z													
⑦	( )·L_z													
⑧	$A_{\alpha\beta}$ = ④+⑤+⑥+⑦													
⑨	( )·T_h													
⑩	( )·P_h													
⑪	$A_{\beta h}$ = ⑨ + ⑩													
⑫	( )·T_α													
⑬	( )·P_α													
⑭	( )·T_h													
⑮	( )·P_h													
⑯	$A_{\beta\alpha}$ = ⑫+⑬+⑭+⑮													
⑰	( )·T_β													
⑱	( )·P_β													
⑲	( )·T_z													
⑳	( )·P_z													
㉑	$A_{\beta\beta}$ = ⑰+⑱+⑲+⑳													

$c - e$ =

e =

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WING FLUTTER CALCULATION

BENDING-TORSION-AILERON

TABLE 5

$\pi\rho =$ $\quad S_\alpha =$

$M =$ $\quad I_\alpha =$

WING DETERMINANT ELEMENTS

$1/k = \frac{v}{b_r\omega} \rightarrow$	( )		( )		( )		( )		( )		( )	
	R	I	R	I	R	I	R	I	R	I	R	I
$\bar{A}_{hh} = 1.0 + ($ $)\cdot A_{hh}$												
$\bar{A}_{h\alpha} = ($ $)+($ $)\cdot A_{h\alpha}$												
$\bar{A}_{\alpha h} = ($ $)+($ $)\cdot A_{\alpha h}$												
$\bar{A}_{\alpha\alpha} = 1.0 + ($ $)\cdot A_{\alpha\alpha}$												

AILERON DETERMINANT ELEMENTS

( )% STATIC BALANCE

$1/k = \frac{v}{b_r\omega} \rightarrow$	( )		( )		( )		( )		( )		( )	
	R	I	R	I	R	I	R	I	R	I	R	I
$\bar{A}_{h\beta} = ($ $)+($ $)\cdot A_{h\beta}$												
$\bar{A}_{\alpha\beta} = ($ $)+($ $)\cdot A_{\alpha\beta}$												
$\bar{A}_{\beta h} = ($ $)+($ $)\cdot A_{\beta h}$												
$\bar{A}_{\beta\alpha} = ($ $)+($ $)\cdot A_{\beta\alpha}$												
$\bar{A}_{\beta\beta} = ($ 1.0 $)+($ $)\cdot A_{\beta\beta}$												
$S_\beta =$												
$I_\beta =$												
$P_{\alpha\beta} =$												

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WING FLUTTER CALCULATION

BENDING-TORSION-AILERON

EVALUATION OF DETERMINANT

TABLE 6

ITEMS INDEPENDENT OF AILERON STATIC BALANCE AND AILERON FREQUENCY

	$1/k = \frac{v}{b_r \omega}$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	
1	$-1/\mu \cdot \bar{A}_{\alpha\alpha}$ = ( ) $\cdot \bar{A}_{hh}$													
2	① $- \bar{A}_{\alpha\alpha}$													
3	$\bar{A}_{hh} \cdot \bar{A}_{\alpha\alpha}$													
4	$\bar{A}_{\alpha h} \cdot \bar{A}_{h\alpha}$													
5	③ − ④													
6	$1/\mu \cdot$ ⑤ = ( ) $\cdot$ ⑤													

ITEMS INDEPENDENT OF AILERON FREQUENCY ( )% AILERON STATIC BALANCE

	$1/k = \frac{v}{b_r \omega}$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	($)_R$	($)_I$	
7	$\bar{A}_{\alpha\beta} \cdot \bar{A}_{\beta\alpha}$													
8	$\bar{A}_{\beta h} \cdot \bar{A}_{h\beta}$													
9	$1/\mu \cdot$ ⑧ = ( ) $\cdot$ ⑧													
10	⑦ + ⑨													
11	$\bar{A}_{\alpha h} \cdot \bar{A}_{h\beta}$													
12	⑪ $\cdot \bar{A}_{\beta\alpha}$													
13	$\bar{A}_{\beta h} \bar{A}_{h\alpha}$													
14	⑬ $\cdot \bar{A}_{\alpha\beta}$													
15	$-\bar{A}_{hh} \cdot$ ⑦													
16	$-\bar{A}_{\alpha\alpha} \cdot$ ⑧													
17	⑫ + ⑭ + ⑮ + ⑯													
18	$1/\mu \cdot$ ⑰ = ( ) $\cdot$ ⑰													

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WING FLUTTER CALCULATION
BENDING-TORSION-AILERON
EVALUATION OF DETERMINANT

TABLE 6 (CONT'D)

ITEMS DEPENDENT ON AILERON FREQUENCY AND STATIC BALANCE: $(\omega_\phi/\omega_\alpha)$=( ); ( )% STATIC BALANCE

$1/k = \frac{v}{b\omega}$ →	(	)	(	)	(	)	(	)	(	)	(	)
	R	I	R	I	R	I	R	I	R	I	R	I
(19) $\bar{A}'_{\phi\phi} = \bar{A}_{\phi\phi}\cdot(\ \ ) + j(\ \ ) = R + jI$		0 †										
(20) $1/\bar{A}'_{\phi\phi} = R/(R^2+I^2) - j\,I/(R^2+I^2)$												
(21) (20)·(10)												
(22) (21)+(2)												
(23) $-\frac{1}{2}\cdot$(22) = λ												
(24) λ^2 = (23)^2												
(25) η = (20)·(18) + (6)												
(26) (24) − (25) = $\lambda^2 - \eta$												
(27) $\sqrt{\lambda^2-\eta}$ = $\sqrt{(26)}$ *												
(28) $\lambda + \sqrt{\lambda^2-\eta}$ = (23)+(27) = $X_1 + jY_1$												
(29) $\lambda - \sqrt{\lambda^2-\eta}$ = (23)−(27) = $X_2 + jY_2$												
(30) $(\omega_\phi/\omega_\alpha)^2\cdot 1/k$ = ( )·1/k												
(31) $\sqrt{X_1}$ = $\sqrt{\text{REAL PART OF (28)}}$												
(32) $\sqrt{X_2}$ = $\sqrt{\text{REAL PART OF (29)}}$												
(33) (30)·(31) = $v/b\omega_1$; [I (28)] ÷ [R (28)] = g												
(34) (30)·(32) = $v/b\omega_2$; [I (29)] ÷ [R (29)] = g												
* SEE TABLE 7												
† FOR 1/k = 0, $\bar{A}'_{\phi\phi} = \bar{A}_{\phi\phi}(\omega_\phi/\omega_\alpha)^2$												

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AUXILIARY WORK SHEET (TABLE V)

SQUARE ROOT OF THE COMPLEX NUMBER $R + jI$

	①	②	③	④	⑤	⑥
	GIVEN	NUMBER				
¼ ↓	R	I	R^2+I^2	$\sqrt{③}$	④ - ①	½ · ⑤

	⑧	⑨	⑦
¼ ↓	2.0 × ⑦	② ÷ ⑧	$\sqrt{⑥}$
		SQUARE	ROOT
	—	R_0	I_0

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APPENDIX II

THE SOLUTION OF FREQUENCY
EQUATIONS BY MATRIX TECHNIQUES

2/5/6

Part A - Example

Consider a typical system of frequency equations of an elastic system vibrating sinusoidally with circular frequency ω :

$$
(1a)\quad
\begin{aligned}
\varphi_1 &= [b_{11}\varphi_1 + b_{12}\varphi_2 + b_{13}\varphi_3 + b_{14}\varphi_4]\,\omega^2 \\
\varphi_2 &= [b_{21}\varphi_1 + b_{22}\varphi_2 + b_{23}\varphi_3 + b_{24}\varphi_4]\,\omega^2 \\
\varphi_3 &= [b_{31}\varphi_1 + b_{32}\varphi_2 + b_{33}\varphi_3 + b_{34}\varphi_4]\,\omega^2 \\
\varphi_4 &= [b_{41}\varphi_1 + b_{42}\varphi_2 + b_{43}\varphi_3 + b_{44}\varphi_4]\,\omega^2
\end{aligned}
$$

Where φ's represent generalized displacements and b_{11}, b_{12}, etc., are numerical constants which are functions of the inertia and elasticity of the system. Their determination is much simpler when actual numbers are used rather than symbols. This will be demonstrated in the worked example which will follow.

At this point the matrix notation is introduced. In this notation, Equation 1a becomes

$$
\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}
=
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{21} & b_{22} & b_{23} & b_{24} \\
b_{31} & b_{32} & b_{33} & b_{34} \\
b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix}
\times
\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} \omega^2
$$

where the "matrix multiplication" indicated on the right between the "square matrix" and the "column matrix" can be readily understood by referring to the conventional form of Equation 1a . This type of multiplication is standard for a matrix of n rows and m columns times another of (necessarily) m rows and p columns, the result being a matrix of n rows and p columns. (Above, the result will be a one-column matrix.)

Since they occur frequently hereafter, examples of matrix multiplication are given here to clarify their meaning. Let M be an arbitrary four-by-four matrix such as

$$M = \begin{bmatrix} 6 & 2 & 1 & -3 \\ 5 & 0 & 7 & 5 \\ 2 & 8 & 6 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix}$$

Postmultiplication of M by an arbitrary column such as

$$C = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

is performed as follows:

$$MC = \begin{bmatrix} 6 & 2 & 1 & -3 \\ 5 & 0 & 7 & 5 \\ 2 & 8 & 6 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 35 \\ 44 \\ 48 \\ 44 \end{bmatrix}$$

where

$$35 = 6 \times 5 + 2 \times 3 + 1 \times 2 - 3 \times 1$$

$$44 = 5 \times 5 + 0 \times 3 + 7 \times 2 + 5 \times 1, \text{ etc.}$$

2/5/6

Premultiplication of M by an arbitrary row

$$R = [3278]$$

is performed as follows:

$$[3278] \begin{bmatrix} 6 & 2 & 1 & -3 \\ 5 & 0 & 7 & 5 \\ 2 & 8 & 6 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix} = [66 \quad 134 \quad 51 \quad 47]$$

where $66 = 3 \times 6 + 2 \times 5 + 7 \times 2 + 8 \times 3$
$134 = 3 \times 2 + 2 \times 0 + 7 \times 8 + 8 \times 9$, etc.

Since both ω^2 and the φ_n are unknown, solutions will give only relative rotations φ_n. Thus there will result, up to a constant multiplier, a solution:

$$\begin{bmatrix} \bar{\varphi}_1 \\ \bar{\varphi}_2 \\ \bar{\varphi}_3 \\ \bar{\varphi}_4 \end{bmatrix}$$

Which may conveniently be "normalized" by dividing all φ values by (say) φ_4, thus getting 1 as the φ_4 entry and values relative to this elsewhere. The term normalize is a convenience adopted from mathematics to suggest the reduction of all values to a common basis for comparison.

Essence of the iteration technique which will be used is to assume such a set of normalized φ values and perform the indicated postmultiplication, on the right of equation (1b.). Normalizing the result gives a new set of φ values. These are used in the same indicated multiplication, the results normalized, and the process continued until two adjacent cycles give insignificant variation in the results.

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The process is then said to converge. That this process does converge (except in unusual cases) is proved in various standard texts on matrices, etc. Iteration is an obvious name for this repeating series of operations.

The final value of $\bar{\phi}_4$ used as a divisor in normalizing is a close approximation to $\frac{1}{\omega^2}$ for the fundamental mode. These techniques are illustrated in the following example.

Assume given the matrix equation:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 1.2 & .9 & .7 & .6 \\ 1.2 & 1.9 & 1.7 & 1.6 \\ 1.2 & 1.9 & 2.7 & 2.6 \\ 1.2 & 1.9 & 2.7 & 3.6 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} \omega^2 \times 10^{-6}$$

The iteration process by postmultiplication on this is given in Table 1.

TABLE I

Calculation of Relative Vibration Amplitudes in First Mode

Assumed Mode	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10
.4	2.60	.3523	2.186	.2901	2.113	.2829	2.100	.2819	2.09793
.6	4.58	.6206	4.672	.6200	4.601	.6161	4.582	.6151	4.57883
.8	6.38	.8645	6.536	.8673	6.468	.8661	6.449	.8658	6.44463
1.0	7.88	1.0000	7.536	1.0000	7.468	1.0000	7.449	1.0000	7.44463

Col.11	Col.12	Col.13	Mode
.281804	2.097683	.281792	.2818
.615651	4.578408	.615040	.6150
.865674	6.444082	.865665	.8657
1.000000	7.444082	1.000000	1.0000

To illustrate the meaning of this Table, the matrix multiplication used to get Column 2 from Column 1 is as follows:

$$\begin{bmatrix} 1.2 & .9 & .7 & .6 \\ 1.2 & 1.9 & 1.7 & 1.6 \\ 1.2 & 1.9 & 2.7 & 2.6 \\ 1.2 & 1.9 & 2.7 & 3.6 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \\ .8 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 2.60 \\ 4.58 \\ 6.38 \\ 7.38 \end{bmatrix}$$

The process involved in going from the second to the third column

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is as follows, correct to four decimal places; $2.60/_{7.38} = .3523$; $4.58/_{7.38} = .6206$; $6.38/_{7.38} = .8645$; $7.38/_{7.38} = 1.$

The entire process is repeated, starting with the third column for postmultiplication, instead of the first, to get successive columns, Each odd-numbered column is a closer approximation to the actual values of φ which define the mode. The frequency equation is

$$10^5/\omega_1^2 = 7.4441$$

from which $\omega_1 = 115.9$ radians/sec., and the frequency is $f_1 = \frac{60}{2\pi}\omega_1 = 1106.8$ cycles/min.

The iteration could have been carried out by premultiplication of the square matrix of Equation (1b) by a row matrix also, but this does not yield the mode; only the frequency is given in this case.

By the nature of the iteration process as set up in the form of Table I there is assurance that this is the fundamental frequency required, and the final column gives the relative displacement values φ for the fundamantal mode.

The method for obtaining higher modes and frequencies is next discussed. No attempt is made here to give the underlying theory, but the technique will be described in sufficient detail to enable the reader to follow each step.

To obtain the second mode from Equation (1b) assume B_1 is the squa[re] matrix on the right side of this equation.
Let

$$\left[r_1, r_2, r_3, r_4\right]$$

be the row matrix obtained by iteration on B_1, by row premultiplication (See Part B for an important modification of this method). Assume this is normalized by dividing the results through by any one of the r's, say r_i. Then the row has a 1 in the i^{th} place and has the form

$$\left[\frac{r_1}{r_i} \quad \frac{r_2}{r_i} \quad \frac{r_3}{r_i} \quad \frac{r_4}{r_i}\right]$$

Let I be the "identity matrix" with unity of the main diagonal and zeros elsewhere:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Let E_1 be the square matrix with the normalized row as its i^{th} row and zeros elsewhere; if $r_i = r_4$ for example, then E is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_1}{r_4} & \frac{r_2}{r_4} & \frac{r_3}{r_4} & \frac{r_4}{r_4} \end{bmatrix} = E_1$$

Form the matrix $I - E_1$; thus, if $r_4 = r_i$;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{r_1}{r_4} & -\frac{r_2}{r_4} & -\frac{r_3}{r_4} & 0 \end{bmatrix} = I - E_1$$

Form the square matrix $B_2 = B_1 (I - E_1)$ by multiplication. The matrix B_2 will now be used exactly as B_1 was used, in the iteration process, and the result (by postmultiplication) will converge on the second mode and frequency ω_2. It may be noted that iteration by premultiplication may be used if only the frequency is desired, thus yielding no information about the mode, however. The process may be repeated to yield the third mode and frequency from a matrix B_3 similarly obtained from B_2, etc.

This process is used in the following to obtain the second and fourth modes and frequencies and the remaining frequency without its mode, in the present example. B_1 is the square matrix previously encountered:

$$B_1 = \begin{bmatrix} 1.2 & .9 & .7 & .6 \\ 1.2 & 1.9 & 1.7 & 1.6 \\ 1.2 & 1.9 & 2.7 & 2.6 \\ 1.2 & 1.9 & 2.7 & 3.6 \end{bmatrix} \times 10^{-5}$$

Iteration by premultiplication on B_1. is performed by first selecting any arbitrary row of numbers and applying the procedure described earlier for premultiplication and iteration. For example,

selecting the arbitrary row $\begin{bmatrix} .5 & .7 & .9 & 1.0 \end{bmatrix}$, the first premultiplication is as follows:

$$\begin{bmatrix} .5 & .7 & .9 & 1.0 \end{bmatrix} \begin{bmatrix} 1.2 & .9 & .7 & .6 \\ 1.2 & 1.9 & 1.7 & 1.6 \\ 1.2 & 1.9 & 2.7 & 2.6 \\ 1.2 & 1.9 & 2.7 & 3.6 \end{bmatrix} = \begin{bmatrix} 3.72 & 5.39 & 6.67 & 7.36 \end{bmatrix}$$

Normalizing the result yields a new row

$$\begin{bmatrix} \frac{3.72}{7.36} & \frac{5.39}{7.36} & \frac{6.67}{7.36} & \frac{7.36}{7.36} \end{bmatrix} = \begin{bmatrix} .506 & .733 & .907 & 1.000 \end{bmatrix}$$

Repeating the premultiplication and normalizing until values of the results converge so that alternate columns show negligible differences, the following row matrix results:

$$\begin{bmatrix} 3.78304 & 5.48164 & 6.74913 & 7.44388 \end{bmatrix}$$

Note that the natural frequency for the first mode is given by the relation $10^5/\omega_1^2 = 7.44388$ (compare with previous solution for ω_1). The figures in the row, however, tell nothing directly about the first mode. Normalizing yields the row matrix

$$\begin{bmatrix} .50821 & .73640 & .90801 & 1.00000 \end{bmatrix}$$

Thus $I-E_1$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -.5082 & -.7364 & -.9080 & 0 \end{bmatrix} = I - E_1$$

and $B_2 = B_1 \; (I - E_1)$ is found by multiplying, row by column, giving the result

$$\begin{bmatrix} .89508 & .45816 & .15520 & 0 \\ .38688 & .72176 & .24720 & 0 \\ -.12132 & -.01464 & .33920 & 0 \\ -.62952 & -.75104 & -.56880 & 0 \end{bmatrix} \times 10^{-5} = B_2$$

Iteration by column postmultiplication on B_2 yields, in the final two columns:

-1.33224	-1.10700
- .96577	- .80249
.20338	.16899
1.20347	1.00000

This gives $\omega_2^2 = 10^5/1.20347 = 83093$, $\omega_2 = 288.26$ radians per second, and $f_2 = 2753$ cycles per minute.

The final column gives the second mode.

Iteration by row premultiplication on B_2 yields the final row

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$$\begin{bmatrix} 1 & .93769 & .44781 & 0 \end{bmatrix}$$

The normalizing this time is done by dividing each time by the <u>first</u> term. Thus E_2 is formed by placing the normalized row in the <u>first</u> row, and $I - E_2$ becomes

$$\begin{bmatrix} 0 & -.93760 & -.44781 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I - E_2$$

Then $B_3 = B_2 (I - E_2)$ is

$$\begin{bmatrix} 0 & -.38107 & -.24563 & 0 \\ 0 & .35902 & .07395 & 0 \\ 0 & .09911 & .39353 & 0 \\ 0 & -.16080 & -.28689 & 0 \end{bmatrix} \times 10^{-5} = B_3$$

Iteration by column postmultiplication would yield the third mode which, though not calculated here, is listed by figures in Table II.

TABLE II

Characteristic Modes for Four-Displacement System

Mode	1st	2nd	3rd	4th
φ_1	.2818	-1.1064	-1.1110	-.5130
φ_2	.6150	-.8024	.7070	1.0000
φ_3	.8657	.1687	1.0000	-.9481
φ_4	1.0000	1.0000	-.8640	.3847

If only the frequency is desired, iteration by row premultiplication on B_3 gives the final rows

$$\begin{bmatrix} 0 & .43949 & .46364 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & .94791 & 1 & 0 \end{bmatrix}$$

Here the normalizing is done by dividing each time by the <u>third</u> term so that $10^5/\omega_3^2 = .46364$, $\omega_3 = 464.42$ and $f_3 = 4435$ cycles per minute. Also $I - E_3$ is formed after placing the normalized row in the <u>third</u> row:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -.94791 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I - E_3$$

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The final matrix, $B_4 = B_3 \ (I - E_3)$, contains only one non-zero column:

$$\begin{bmatrix} 0 & -.14823 & 0 & 0 \\ 0 & .28892 & 0 & 0 \\ 0 & -.27392 & 0 & 0 \\ 0 & .11115 & 0 & 0 \end{bmatrix} \times 10^{-5} = B_4$$

Iteration on this is unnecessary, the immediate result being $\omega_4 = 10^5/.28892 = 346116.5$, $\omega_4 = 588.32$, and $f_4 = 5618$ cycles per minute. This follows because the matrix B_4 would occur in an equation of the form of Equation (1b). In the non-matrix form (Equation 1a), the second equation would be

$$\varphi_2 = .28892 \ \varphi_2 \ \times \omega_4^2 \times 10^{-5}$$

To find the mode, it may be noted that $\varphi_1 = -.14823/.28892 = -.5130$; $\varphi_2 = 1.0000$; $\varphi_3 = -.27392/.28892 = -.9481$; $\varphi_4 = .11115/.28892 = .3847$

This mode is included in Table II.

Part B - Discussion of Non-Dominant Latent Roots of a Lambda Matrix for Elastic System

It has been demonstrated that when the dominant root of a "lambda matrix" (i.e. typical frequency matrix as discussed in Part A) has been found, the remaining roots can be obtained successively in order of the moduli (absolute values), by the construction of auxiliary matrices which contain all the latent roots except those already obtained. If λ_1, is the dominant root (first root found) and K_1 is the matrix row obtained by iterative premultiplication of the matrix $[U]$ and if K_{r1} be any non-zero element (number) in K_1, then the matrix E (See Part A) can be defined as that square matrix which has K_1/K_{r1} for its r^{th} row and its remaining n-1 rows null. The matrix $[V]$ which contains all the latent roots except the dominant root is then

$$[V] = [U] \quad [I - E]$$

In general, then, in order to obtain the higher roots by this method, it is necessary to obtain a row K by iterative premultiplication. However, for the special case of a conservative system oscillating in simple harmonic motion, it can be shown that if the fundamental mode is known, the matrix K can be obtained without resorting to iterative premultiplication.

An important property of any lambda matrix is: if k_r be the modal column for the root λ_r and K_s be the matrix row obtained by iterative premultiplication (modal row) for any other root λ_s then

$$[K_s][k_r] = 0 \qquad (r \neq s)$$

This is known as the generalized orthogonality condition for lambda matrices.

For the case of the lambda matrix of a conservative system the generalized orthogonality condition reduces to a simple form. As an example of such a system, consider the case of bending of a cantilevered vibrating elastic structure. For this case it can easily

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be shown (as consequence of Maxwell's Reciprocal Theorem) that:

$$③ \quad \sum_{1}^{n} m_i k_{ri} k_{si} = 0$$

where

m_i = generalized mass "acting" at station i
k_{ri} = maximum displacement at station i when structure vibrates in r^{th} normal mode
k_{si} = maximum displacement at station i when structure vibrates in s^{th} normal mode

Equation 3 is usually referred to as the orthogonality condition for normal modes of the structure. In matrix notation ③ becomes:

$$[m k_r][k_s] = 0$$

Now since for any lambda matrix $[K_r][k_s] = 0$, then $[m k_r][k_s]$

$= [K_r][k_s]$ (up to a constant factor) or ④ $[m k_r] = [K_r]$

(up to a constant factor). Now if any non-zero element in $[m k_r]$ is equal to the same element in $[K_r]$ then each element in $[m k_r]$ is equal to the corresponding element of $[K_r]$, i.e., if

$$[a_1 \ a_2 \ ---- \ a_n] = [b_1 \ b_2 \ ---- b_n] \text{ (up to a constant factor)}$$

and if $a_n = b_n$, then $a_1 = b_1$, $a_2 = b_2$, etc.

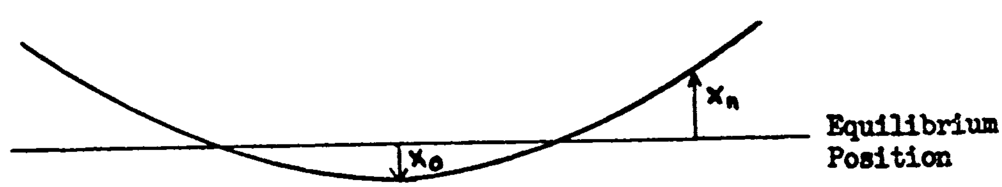
Thus, if the matrix $[K]$ is normalized by dividing through by any of its non-zero elements and the matrix $[m k]$ is also normalized using the element in the same position as for $[K]$, then the two are equal.

Thus is furnished a means of calculating the result of iterative premultiplication for elastic systems without iterative premultiplication.

Part C - Computation of Higher Vibration Modes of A Conventional Airplane Wing

The orthogonality condition for the normal modes of an elastic structure (See Parts A and B) can easily be applied to determining the higher modes and frequencies of an airplane wing of conventional configuration. The application extends equally well to the determination of both coupled and uncoupled higher modes. If the wing be considered a "free-free" system the symmetric and unsymmetric higher coupled and uncoupled modes can be obtained by setting up the matrix [E] (Equation 1), by a simple algebraic operation involving the orthogonality condition (for the specific case in question) and the unknown modal column. In the following sections the elements of the matrix [E] are determined explicitly for each case.

Symmetric Bending of a Free-Free Wing



If the displacement from the equilibrium position at station i is given by x_i for the fundamental mode, and if y_i is the displacement at i in the second mode, then orthogonality condition is

$$⑤ \quad \sum_0^n m_i x_i y_i = 0 \quad \text{or} \quad m_o x_o y_o + m_1 x_1 y_1 + \dots + m_n x_n y_n = 0$$

When the system is oscillating in the second mode the balance condition i

$$⑥ \quad \sum_0^n m_i y_i = 0 \quad \text{or} \quad m_o y_o + m_1 y_1 + \dots\dots + m_n y_n = 0$$

Now y_o can be eliminated between ⑤ and ⑥ by multiplying equation ⑥ by x_o and subtracting the resulting equation from ⑤ ; then

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$$m_1 \ (x_1 - x_o) \ y_1 \ + \ m_2 \ (x_2 - x_o) + \ldots + m_n \ (x_n - x_o) = 0$$

(7) Let the last station be used for normalizing; then if

(8) $$a_{n1} = \frac{m_1 \ (x_1 - x_o)}{m_n \ (x_n - x_o)} ; \quad a_{n2} = \frac{m_2 \ (x_2 - x_o)}{m_n \ x_n - x_o)} ; \quad \ldots ; \quad a_{nn} = 1.$$

the matrix E becomes

(9) $$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & & 1 \end{bmatrix}$$

and $[V] = [U] [I - E]$ is the basic matrix to be iterated by post-multiplication to determine the succeeding modo.

Unsymmetric bending

The orthogonality condition is the same as that for symmetric bending, i.e., $\sum m_1 \ x_1 \ y_1 = o$; however, in the unsymmetric case the deflection at station zero must always be zero; therefore, following (8) with $x_o = 0$ we can immediately write for this case

(10) $$a_{n1} = \frac{m_1 \ x_1}{m_n \ x_n} ; \quad a_{n2} = \frac{m_2 \ x_2}{m_n \ x_n} ; \quad a_{n3} = \frac{m_3 \ x_3}{m_n \ x_n} ; \ --- \ a_{nn} = 1;$$

and $[E]$ is of the same form as (9).

Torsion

Since the fuselage pitching moment of inertia is usually very large compared to wing pitching moments of inertia the uncoupled torsion modes are usually considered to be cantilevered modes. For cantilevered torsion the orthogonality condition is

$$\sum_{i}^{n} I_i \theta_i \alpha_i = 0$$

Where I_i is the mass moment of inertia about the elastic axis at station i; α_i is the maximum torsional displacement about the elastic axis at

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station i in the fundamental mode, and Θ_i is the corresponding displacement at i in the second normal mode.

(11) $a_{n1} = \frac{I_1 \alpha_i}{I_n \alpha_n}$; $a_{n2} = \frac{I_2 \alpha_2}{I_n \alpha_n}$; .... ; $a_{nn} = 1$

Coupled Modes (See AppendixIII)

The orthogonality condition for coupled modes is

(12) $$\sum_{0}^{n} \left[m_i x_i y_i + I_i \alpha_i \Theta_i + S_i (x_i \Theta_i + y_i \alpha_i) \right] = 0$$

where x_i is maximum bending deflection of elastic axis at station i in normal mode (a)

α_i is maximum torsional deflection about elastic axis at station i in normal mode (a)

y_i is maximum bending deflection of elastic axis at station i in normal mode (b)

Θ_i is maximum torsional deflection about elastic axis at station i in normal mode (b)

S_i is static mass moment about elastic axis

Then

(13) $$(m_0 x_0 + s_0 \alpha_0) y_0 + (m_1 x_1 + s_1 \alpha_1) y_1 + \dots + (m_n x_n + s_n \alpha_n) y_n + $$
$$+ (I_0 \alpha_0 + s_0 x_0) \Theta_0 + \dots + (I_n \alpha_n + s_n x_n) \Theta_n = 0$$

is equivalent to (12).

Balance Conditions: symmetric coupled modes

$$\sum_{0}^{n} F_i = 0 = \sum_{0}^{n} (m_i y_i + s_i \Theta_i)$$
$$\sum_{0}^{n} M_i = 0 = \sum_{0}^{n} (s_i y_i + I_i \Theta_i)$$

where F_i is vertical force at i and M_i is torsional moment at i

Hence

(14) $$m_0 y_0 + m_1 y_1 + \dots + m_n y_n + s_0 \Theta_0 + \dots + s_n \Theta_n = 0$$
(15) $$s_0 y_0 + s_1 y_1 + \dots + s_n y_n + I_0 \Theta_0 + \dots + I_n \Theta_n = 0$$

Multiply (14) by x_0 and (15) by α_0 and add results; then

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(16) $$(m_o\ x_o + s_o\alpha_o)\,y_o + (m_1\,x_o + s_1\alpha_1)\,y_1 + \dots + (m_n\,x_o + s_n\alpha_n)\,y_n$$
$$(S_o\ x_o + I_o\alpha_o)\,\Theta_o + \dots + (s_n\,x_o + I_n\alpha_o)\,\Theta_n = 0$$

By subtracting (16) from (13) y_o and Θ_o are eliminated. The desired result is

$$\left[m_1\ (x_1 - x_o) + s_1\,(\alpha_1 - \alpha_o)\right] y_1 + \left[m_2\ (x_2 - x_o) + s_2\ (\alpha_2 - \alpha_o)\right] y_2$$
$$+ \dots + m_n\,(x_n - x_o) + s_n\,(\alpha_n - \alpha_o)\big]\, y_n + \left[s_1{}_0\,(x_1 - x_o) + I_1\,(\alpha_1 - \alpha_o)\right]\Theta_1$$
$$+ \dots + \left[s_n\,(x_n - x_o) + I_n\ (\alpha_n - \alpha_o)\right]\Theta_n = 0$$

If the coefficients of $y_1 \dots y_n$ and $\Theta_1 \dots \Theta_n$ are selected and normalized on the $_n$th (last) element, the results are

$$a_{n1} = \frac{m_1\,(x_1 - x_o) + s_1\,(\alpha_1 - \alpha_o)}{m_n\,(x_n - x_o) + s_n\,(\alpha_n - \alpha_o)}\ ; \quad a_{n2} = \frac{m_2\,(x_2 - x_o) + s_2\,(\alpha_2 - \alpha_o)}{m_n\,(x_n - x_o) + s_n\,(\alpha_n - \alpha_o)}\ ;$$

$$a_{nn} = 1;\quad a_{n,\,n+1} = \frac{s_1\,(x_1 - x_o) + I_1\ (\alpha_1 - \alpha_o)}{m_n\,(x_n - x_o) + s_n\ (\alpha_n - \alpha_o)};$$

$$\dots;\ a_{n,2n} = \frac{s_n\,(x_n - x_o) + I_n\ (\alpha_n - \alpha_o)}{m_n\,(x_n - x_o) + s_n\ (\alpha_n - \alpha_o)}$$

The matrix E in this case is of the form

(19) $$\begin{bmatrix} 0 & 0 \dots & 0 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \dots & 0 & 0 \dots & 0 \\ a_{n1} & a_{n2} & 1 & a_{n,(n+1)} & a_{n,2n} \\ 0 & 0 \dots & 0 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Unsymmetric Coupled Modes

Since for unsymmetric modes Θ_o, α_o, x_o and y_o are always zero, the elements $a_{n1} \dots a_{n,2n}$ can be expressed immediately as

$$a_{n1} = \frac{m_1\,x_1 + s_1\,\alpha_1}{m_n\,x_n + s_n\,\alpha_n}\ ; \qquad a_{n2} = \frac{m_2\,x_2 + s_2\,\alpha_2}{m_n\,x_n + s_n\,\alpha_n}\ ; \dots ;\ a_{nn} = 1\ ;$$

$$a_{n;\,n+1} = \frac{s_1\,x_1 + I_1\ \alpha_1}{m_n x_n \quad s_n\,\alpha_n};\qquad a_{n;\,2n} = \frac{s_n\,x_n + I_n\,\alpha_n}{m_n\,x_n + s_n\,\alpha_n}$$

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APPENDIX III

COUPLED MODES OF VIBRATION OF A FREE-FREE WING IN AIR

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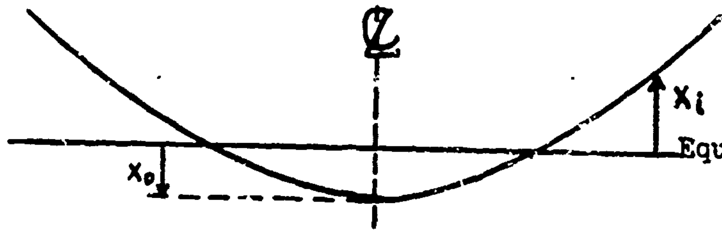

SYMMETRIC MODES

Let $x_1, x_2, \ldots x_n$ be the total bending deflections, from the equilibrium position, of the elastic axis, at stations 1, 2, ... n, Let x_o be the displacement of the centerline (station zero). Then the displacement of the elastic axis at station i relative to the centerline is $x_i - x_o$.

If $a_{11}, a_{12} \ldots a_{nn}$ be the bending influence coefficients, then the deflection relative to the centerline, at any station i, due to forces $F_1, F_2 \ldots F_n$ acting at stations 1, 2, ... n, respectively, can be written as: $x_i - x_o = a_{i1} F_1 + a_{i2} F_2 + \ldots + a_{in} F_n$

or

$$
\begin{aligned}
x_1 - x_o &= a_{11} F_1 + a_{12} F_2 + \ldots\ldots + a_{1n} F_n \\
x_2 - x_o &= a_{21} F_1 + a_{22} F_2 + \ldots\ldots + a_{2n} F_n \\
&\vdots \\
x_n - x_o &= a_{n1} F_1 + a_{n2} F_2 + \ldots\ldots + a_{nn} F_n
\end{aligned}
\qquad (1)
$$

A set of equations similar to 1 can be written for the torsional (angular) deflections

$$
\begin{aligned}
\alpha_1 - \alpha_o &= b_{11} M_1 + b_{12} M_2 + \ldots\ldots + b_{1n} M_n \\
\alpha_2 - \alpha_o &= b_{21} M_1 + b_{22} M_2 \ldots\ldots + b_{2n} M_n \\
&\vdots \\
\alpha_n - \alpha_o &= b_{n1} M_1 + b_{n2} M_2 + \ldots\ldots + b_{nn} M_n
\end{aligned}
\qquad (2)
$$

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where

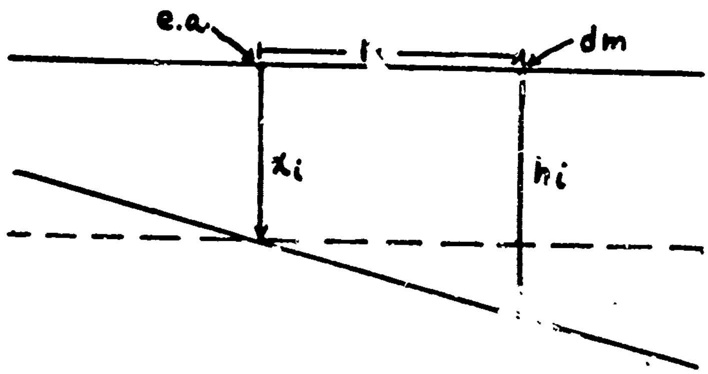
α_i = angular displacement of wing at station i, about the elastic axis

b_{ii} = torsional influence coefficient

M_i = torsional moment acting at station i

Equations (1) and (2) give the deflections under any load. Now if the system is oscillating in simple harmonic motion about the equilibrium position, with frequency ω and if the maximum displacement of an element of mass dm is h_i, then the inertia force due to this element of mass is:

$$dF = \omega^2 h_i \, dm = \omega^2 (x_i + r_i \alpha_i) \, dm \quad (3)$$



The inertia moment about the elastic axis is:

$$dM = \omega^2 r_i \, h_i \, dm = \omega^2 (r_i \, x_i \, dm + r_i^2 \, \alpha_i \, dm) \quad (4)$$

Then the total inertia force and moment at station i, is obtained by integrating chordwise.

$$F_i = \omega^2 \left[x_i \int dm_i + \alpha_i \int r_i \, dm_i \right] = \omega^2 \left[M_i \, x_i + S_i \alpha_i \right] \quad (5)$$

$$M_i = \omega^2 \left[x_i \int r_i dm + \alpha_i \int r_i^2 \, dm \right] = \omega^2 \left[S_i \, x_i + I_i \alpha_i \right] \quad (6)$$

where

M_i = total mass at station i

S_i = static moment about e. a., at station i

I_i = mass moment of inertia about e.a., at station i

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It should be noted at this point that the analysis which involves the inertia of the wing alone is strictly speaking applicable to vibration in a vacuum. In order to determine the coupled frequencies in air, it is necessary to consider the "inertia" force of the air oscillating with the structure.

From flutter theory (See Appendix IV) it can be shown that the inertia force and moment about the elastic axis, for an oscillating airfoil section per unit length of span, is:

$$dF_i = \frac{\pi \rho c_i^2}{4}\left(-\ddot{x}_i + s_i \ddot{\alpha}_i\right)$$
$$dM_i = \frac{\pi \rho c_i^2}{4}\left(s_i \ddot{x}_i - \left[\frac{c_i^2}{32} + s_i^2\right]\ddot{\alpha}_i\right) \qquad (7)$$

where

C_i = total chord length at station i

s_i = distance between e.a. and midchord at station i (+ for e.a. aft of midchord)

ρ = density of air

for harmonic motion:

$$\ddot{x}_i = -\omega^2 x_i \qquad (8)$$
$$\ddot{\alpha}_i = -\omega^2 \alpha_i$$

substituting in (7)

$$dF_i = \frac{\pi \rho \omega^2 c_i^2}{4}\left[x_i - s_i \alpha_i\right]$$
$$dM_i = \frac{\pi \rho \omega^2 c_i^2}{4}\left[-s_i x_i + \left(\frac{c_i^2}{32} + s_i^2\right)\alpha_i\right] \qquad (9)$$

Then the total force and moment on an element of length Δy_i along the span is : (inertia of air only)

$$F_i = \frac{\pi \rho c_i^2 \Delta y_i}{4}\left[x_i - s_i \alpha_i\right]$$
$$M_i = \frac{\pi \rho c_i^2 \Delta y_i}{4}\left[-s_i x_i + \left(\frac{c_i^2}{32} + s^2\right)\alpha_i\right]$$

where C_i, s_i x_i and α_i are the average values in the interval Δy_i

The total inertia force and moment (structure + air) acting at

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station i is:

$$F_i = \omega^2\left[\left(m_i + \frac{\pi\rho c_i^2 \Delta y_i}{4}\right)x_i + \left(S_i - \frac{\pi\rho c_i^2 \Delta y_i a_i}{4}\right)\alpha_i\right]$$

$$= \omega^2\left[\bar{m}_i x_i + \bar{S}_i \alpha_i\right] \qquad (11)$$

$$M_i = \omega^2\left[\left(S_i - \frac{\pi\rho c_i^2 \Delta y_i a_i}{4}\right)x_i + \alpha_i\left(I_i + \frac{\pi\rho c_i^4 \Delta y_i}{128} + \frac{\pi\rho c_i^2 a_i \Delta y_i}{4}\right)\right]$$

$$= \omega^2\left[\bar{S}_i x_i + \bar{I}_i \alpha_i\right] \qquad (12)$$

Substituting (11) and (12) in equations (1) and (2), equations (13) which are $2n$ homogeneous linear simultaneous equations, are obtained.

$$x_1 - x_0 = \omega^2\left[\bar{m}_1 a_{11} x_1 + \bar{m}_2 a_{12} x_2 + \cdots \bar{m}_n a_{1n} x_n + \bar{S}_1 a_{11}\alpha_1 + \bar{S}_2 a_{12}\alpha_2 + \cdots \bar{S}_n a_{1n}\alpha_n\right]$$

$$x_2 - x_0 = \omega^2\left[\bar{m}_1 a_{21} x_1 + \bar{m}_2 a_{22} x_2 + \cdots \bar{m}_n a_{2n} x_n + \bar{S}_1 a_{21}\alpha_1 + \cdots + \bar{S}_n a_{2n}\alpha_n\right]$$

$$\alpha_1 - \alpha_0 = \omega^2\left[\bar{S}_1 b_{11} x_1 + \bar{S}_2 b_{12} x_2 + \cdots \bar{S}_n b_{1n} x_n + \bar{I}_1 b_{11}\alpha_1 + \bar{I}_2 b_{12}\alpha_2 + \cdots \bar{I}_n b_{1n}\alpha_n\right]$$

$$\alpha_n - \alpha_0 = \omega^2\left[\bar{S}_1 b_{n1} x_1 + \bar{S}_2 b_{n2} x_2 + \cdots + \bar{S}_n b_{nn} x_n + \bar{I}_1 b_{n1}\alpha_1 + \cdots \bar{I}_n b_{nn}\alpha_n\right]$$

Now for symmetric vibrations each half of the wing must be in equilibrium

$$\sum F = 0$$

$$\sum M = 0$$

That is: (since $\omega \neq 0$)

$$\bar{m}_0 x_0 + \bar{m}_1 x_1 + \cdots \bar{m}_n x_n + \bar{S}_0 \alpha_0 + \bar{S}_1 \alpha_1 + \cdots \bar{S}_n \alpha_n = 0 \qquad (14)$$

$$\bar{S}_0 x_0 + \bar{S}_1 x_1 + \cdots \bar{S}_n x_n + \bar{I}_0 \alpha_0 + \bar{I}_1 \alpha_1 + \cdots \bar{I}_n \alpha_n = 0 \qquad (15)$$

x_0 and α_0 can be eliminated from (13) by using (14) and (15)

This is usually accomplished by multiplying the first equation in (13)

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by $\bar{m}_1$, the second by $\bar{m}_2$ etc. To the n^{th} by $\bar{m}_n$, the $(n+1)$ by $\bar{S}_1$, $(n+2)$ by $\bar{S}_2$ etc., adding the 2n equations the following is obtained:

$$\bar{m}_1 x_1 + \bar{m}_2 x_2 \dots + \bar{m}_n x_n + \bar{S}_1\alpha_1 + \bar{S}_2\alpha_2 + \dots \bar{S}_n\alpha_n - x_0\sum_1^n \bar{m}_i - \alpha_0 \sum_1^n \bar{S}_i = \quad (16)$$

$$\omega^2 \left[A_1 x_1 + A_2 x_2 + \dots + A_n x_n + B_1\alpha_1 + B_2\alpha_2 + \dots + B_n\alpha_n\right]$$

Where A_i is coefficient of x_i } etc.
B_i is coefficient of α_i

Now from (14)

$$-\left(\bar{m}_0 x_0 + \bar{S}_0 \alpha_0\right) = \bar{m}_1 x_1 + \bar{m}_2 x_2 + \dots \bar{m}_n x_n + \bar{S}_1\alpha + \dots + \bar{S}_n\alpha_n$$

Therefore, the left side of (16) becomes

$$-x_0 \sum_0^n \bar{m}_i - \alpha_0 \sum_0^n \bar{S}_i$$

and if $\bar{M}$ = mass of $\frac{1}{2}$ of airplane (+ air moving with it)
$\bar{S}$ = static moment about e.a. of $\frac{1}{2}$ of airplane (+ air moving with it)

Then: $$-\bar{M} x_0 - \bar{S}_{\alpha_0} = \omega^2 \left[A_1 x_1 + A_2 x_2 + \dots A_n x_n + B_1\alpha_1 + \dots B_n\alpha_n\right] \quad (17)$$

Another expression in x_0 and α_0 can be obtained by multiplying equation 1 of (13) by $\bar{S}_1$, second by $\bar{S}_2$, n^{th} by $\bar{S}_n$, $(n+1)$ by $\bar{I}_1$ etc. to $(2n)$ by $\bar{I}_n$.

Adding these equations:

$$\bar{S}_1 x_1 + \bar{S}_2 x_2 + \dots \bar{S}_n x_n + \bar{I}_1\alpha_1 + \bar{I}_2\alpha_2 + \dots + \bar{I}_n\alpha_n - x_0\sum_1^n \bar{S}_i - \alpha_0\sum_1^n \bar{I}_i$$

$$= \omega^2 \left[C_1 x_1 + C_2 x_2 + \dots + C_n x_n + D_1\alpha_1 + \dots + D_n\alpha_n\right]$$

where

C_i is coefficient of x
D_i is coefficient of α

from (15)

$$-\bar{S}_0 x_0 - \bar{I}_0\alpha_0 = \bar{S}_1 x_1 + \bar{S}_2 x_2 + \dots \bar{S}_n x_n + \bar{I}_1\alpha_1 + \bar{I}_2\alpha_2 \dots \bar{I}_n\alpha_n \qquad (19)$$

Therefore, left side of (18) becomes

$$-x_0\sum_0^n \bar{S}_i - \alpha_0 \sum_0^n \bar{I}_i$$

and if $\bar{I}$ pitching mass moment of inertia of $\frac{1}{2}$ of airplane about e.a.

(plus aerodynamic effects)

then

$$-\bar{S}x_0 - \bar{I}\alpha_0 = \omega^2\left[C_1 x_1 + C_2 x_2 + \dots + C_n x_n + D_1\alpha_1 + D_2\alpha_2 + \dots D_n\alpha_n\right]$$

from (17) and (19) we can solve for x_0 and α_0 in terms of x_1, x_2 ... x_n, α_1, α_2, ... α_n and ω. Then x_0 and α_0 will be of the form:

$$x_0 = \omega^2\left[E_1 x_1 + E_2 x_2 + \dots E_n x_n + G_1\alpha_1 + \dots + G_n\alpha_n\right]$$

$$\alpha_0 = \omega^2\left[H_1 x_1 + H_2 x_2 + \dots H_n x_n + J_1\alpha_1 + \dots + J_n\alpha_n\right] \qquad (20)$$

Substituting in (13) for x_0 and α_0 the resulting equations are of the form:

$$x_1 = \omega^2\left[e_{11}x_1 + e_{12}x_2 + \dots e_{1n}x_n + f_{11}\alpha_1 + f_{12}\alpha_2 + \dots f_{1n}\alpha_n\right]$$

$$\vdots$$

$$x_n = \omega^2\left[e_{n1}x_1 + e_{n2}x_2 + \dots e_{nn}x_n + f_{n1}\alpha_1 + \dots\dots f_{nn}\alpha_n\right]$$

$$\alpha_1 = \omega^2\left[g_{11}x_1 + g_{12}x_2 + \dots g_{1n}x_n + k_{11}\alpha_1 + \dots\dots k_{1n}\alpha_n\right]$$

$$\vdots$$

$$\alpha_n = \omega^2\left[g_{n1}x_1 + g_{n2}x_2 + \dots g_{nn}x_n + k_{11}\alpha_1 + \dots\dots k_{nn}\alpha_n\right]$$

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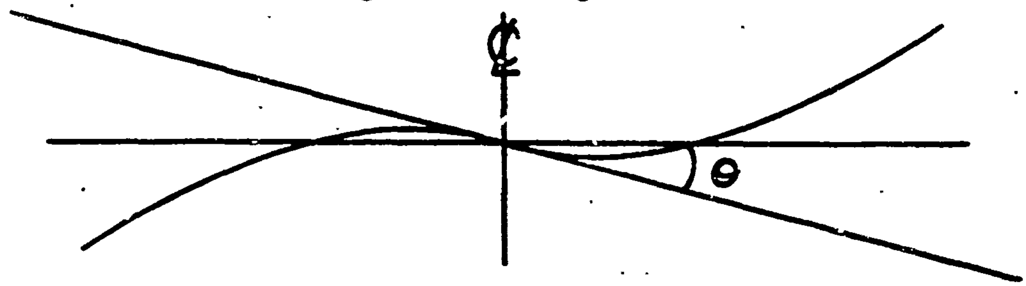
The above set of 2n linear homogeneous simultaneous equations in 2n + 1 unknowns can best be solved by an iteration process. The method generally used is identical with the one used in obtaining the uncoupled modes and frequencies by matrix iteration.

Anti-Symmetric Modes

For the case of anti-symmetric motion the wing elastic axis at the centerline does not deflect vertically. However, the fuselage rotates through some angle Θ and the tangent to the elastic axis at the centerline rotates through the same angle.



If y_i is the distance from ℄ to station i, then the deflection of the elastic axis, from the neutral position, x_i, with respect to the tangent line is $x_i - y_i\Theta$. The deflection can then be expressed as a function of the influence coefficients and forces acting on the system.

$$x_i - y_i\Theta = a_{i1}F_1 + a_{i2}F_2 \cdots\cdots + a_{in}F_n \qquad (22)$$

For anti-symmetric torsion the angular deflection at the ℄ is zero. The torsional deflections can then be expressed as:

$$\alpha_i = b_{i1}M_1 + b_{i2}M_2 + \cdots\cdots b_{in}M_n \qquad (23)$$

The necessary balance condition for $\frac{1}{2}$ of the total system is that the rolling moments for each half be zero. The balance condition then is:

$$I_0\Theta + \bar{M}_1x_1y_1 + \bar{M}_2x_2y_2 + \cdots\cdots + \bar{M}_nx_ny_n + \bar{S}_1y_1\alpha_1 + \cdots\cdot \bar{S}_ny_n\alpha_n = 0 \qquad (24)$$

where I_0 is the rolling moment of inertia of $\frac{1}{2}$ fuselage and tail surfaces and effect of air moving with the structure. Equation (24) is used to eliminate Θ from equation (22). The analysis for the anti-symmetric modes from this point on is similar to the case for the symmetric modes.

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Orthogonality Condition-Coupled Modes

Let

x_{i1} = Bending deflection from equil orium position in the i^{th} coupled mode, at station i. (Second subscript denotes mode)

x_{ij} = coupled deflection at station i, in j^{th} mode

α_{ii}, α_{ij} = torsional deflection in i^{th} and j^{th} modes at station i

$\lambda_i = \frac{1}{\omega_i^2}$

Then equation 13 for i^{th} mode can be written

$$\lambda_i(x_{1i} - x_{oi}) = (\bar{m}_1 x_{1i} + \bar{S}_1 \alpha_{1i}) a_{11} + \cdots\cdots + (m_n x_{ni} + \bar{S}_n \alpha_{ni}) a_{1n}$$

$$\vdots$$

$$\lambda_i(x_{ni} - x_{oi}) = (\bar{m}_1 x_{1i} + \bar{S}_1 \alpha_{1i}) a_{n1} + \cdots\cdots + (\bar{m}_n x_{ni} + \bar{S}_n \alpha_{ni}) a_{nn} \quad (25)$$

$$\lambda_i(\alpha_{1i} - \alpha_{oi}) = (\bar{S}_1 x_{1i} + \bar{I}_1 \alpha_{1i}) b_{11} + \cdots\cdots + (\bar{S}_n x_{ni} + \bar{I}_n \alpha_{ni}) b_{1n}$$

$$\vdots$$

$$\lambda_i(\alpha_{ni} - \alpha_{oi}) = (\bar{S}_1 x_{1i} + \bar{I}_1 \alpha_{1i}) b_{n1} + \cdots\cdots + (\bar{S}_n x_{ni} + \bar{I}_n \alpha_{ni}) b_{nn}$$

Similarly for the j mode

$$\lambda_j(x_{1j} - x_{oj}) = (\bar{m}_1 x_{1j} + \bar{S}_1 \alpha_{1j}) a_{11} + \cdots\cdots + (\bar{m}_n x_{nj} + \bar{S}_n \alpha_{nj}) a_{1n}$$

$$\vdots$$

$$\lambda_j(x_{nj} - x_{oj}) = (\bar{m}_1 x_{1j} + \bar{S}_1 \alpha_{1j}) a_{n1} + \cdots\cdots + (\bar{m}_n x_{nj} + \bar{S}_n \alpha_{nj}) a_{nn} \quad (26)$$

$$\lambda_j(\alpha_{1j} - \alpha_{oj}) = (\bar{S}_1 x_{1j} + \bar{I}_1 \alpha_{1j}) b_{11} + \cdots\cdots + (\bar{S}_n x_{nj} + \bar{I}_n \alpha_{nj}) b_{1n}$$

$$\vdots$$

$$\lambda_j(\alpha_{nj} - \alpha_{oj}) = (\bar{S}_1 x_{1j} + \bar{I}_1 \alpha_{1j}) b_{n1} + \cdots\cdots + (\bar{S}_n x_{nj} + \bar{I}_n \alpha_{nj}) b_{nn}$$

in equation (25) multiply first equation by $(\bar{m}_1 x_{1j} + \bar{S}_1 \alpha_{1j})$

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multiply second by $(\bar{m}_2 x_{2j} + \bar{S}_2 \alpha_{2j})$ etc. to n^{th} equation by $(\bar{m}_n x_{nj} + \bar{S}_n \alpha_{nj})$ Multiply equation $n + 1$ by $(\bar{S}_1 x_{1j} + \bar{I}_1 \alpha_{1j})$ etc. to equation 2_n by $(\bar{S}_n x_{nj} + \bar{I}_n \alpha_{nj})$

Let the new set of equations be denoted as equations (27).

In equations (26) multiply first by $(\bar{m}_1 x_{1i} + \bar{S}_1 \alpha_{1i})$, n^{th} by $(\bar{m}_n x_{ni} + \bar{S}_n \alpha_{ni})$, $(n+1)$ by $(\bar{S}_1 x_{1i} + \bar{I}_1 \alpha_{1i})$. $2n^{th}$ by $(\bar{S}_n x_{ni} + \bar{I}_n \alpha_{ni})$. Let these equations be denoted as equations (28).

Add equations (27) and obtain $\sum$ (27)

Add equations (28) and obtain $\sum$ (28)

Then $\sum$ (28) $-$ $\sum$ (27) yields zero on the right side of the equation.

$$\lambda_j \Big\{ \sum_1^n \bar{m}_i x_{ii} x_{ij} - x_{oi} \sum_1^n \bar{m}_i x_{ij} + \sum_1^n \bar{S}_i x_{ii} \alpha_{ij} - x_{oi} \sum_1^n \bar{S}_i \alpha_{ij} + \sum_1^n \bar{I}_i \alpha_{ii} \alpha_{ij} - \alpha_{oi} \sum_1^n \bar{I}_i \alpha_{ij} + \sum_1^n \bar{S}_i x_{ij} \alpha_{ii} - \alpha_{oi} \sum_1^n \bar{S}_i x_{ij} \Big\} \quad (29)$$

$$- \lambda_i \Big\{ \sum_1^n \bar{m} x_{ii} x_{ij} - x_{oj} \sum_1^n \bar{m}_i x_{ii} + \sum_1^n \bar{S}_i x_{io} \alpha_{ii} - x_{oj} \sum_1^n \bar{S}_i \alpha_{ii} + \sum_1^n \bar{I}_i \alpha_{ii} \alpha_{ij} - \alpha_{oj} \sum_1^n \bar{I}_i \alpha_{ii} + \sum_1^n \bar{S}_i x_{ii} \alpha_{ij} - \alpha_{oj} \sum_1^n \bar{S}_i x_{ii} \Big\} = 0$$

or

$$\lambda_j \Big\{ \sum^n \bar{m}_i x_{ii} x_{ij} + \sum_o^n \bar{S}_i x_{ii} \alpha_{ij} + \sum_o^n \bar{I}_i \alpha_{ii} \alpha_{ij} + \sum_o^n \bar{S}_i x_{ij} \alpha_{ii} - x_{oi} \Big[\sum_o^n \bar{m}_i x_{ij} + \sum_o^n \bar{S}_i \alpha_{ij} \Big] - \alpha_{oi} \Big[\sum_o^n \bar{I}_i \alpha_{ij} + \sum_o^n \bar{S}_i x_{ij} \Big] \Big\} \quad (30)$$

$$- \lambda_i \Big\{ \sum_o^n \bar{m}_i x_{ii} x_{ij} + \sum_o^n \bar{S}_i x_{ii} \alpha_{ij} + \sum_o^n I_i \alpha_{ii} \alpha_{ij} + \sum_o^n \bar{S}_i x_{ij} \alpha_{ii} - x_{oj} \Big[\sum_o^n m_i x_{ii} + \sum_o^n \bar{S}_i \alpha_{ii} \Big] - \alpha_{oj} \Big[\sum_o^n \bar{I}_i \alpha_{ii} + \sum_o^n \bar{S} x_{ii} \Big] \Big\} = 0$$

Now by (14) and (15) the coefficient of x_{oi}, α_{oi}, x_{oj} and α_{oj} are each zero in equation (30)

$$\therefore (\lambda_j - \lambda_i) \Big\{ \sum_o^n (\bar{m}_i x_{ii} x_{ij} + \bar{S}_i x_{ii} \alpha_{ij} + \bar{S}_i x_{ij} \alpha_{ii} + \bar{I}_i \alpha_{ii} \alpha_{ij}) \Big\} = 0 \quad (31)$$

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Now since by hypothesis $\omega_i \neq \omega_j$, $\lambda_i \neq \lambda_j$ then:

$$\sum_{0}^{n} (\bar{m}_i x_{ii} x_{ij} + \bar{S}_i x_{ii} \alpha_{ij} + \bar{S}_i x_{ij} \alpha_{ii} + \bar{I}_i \alpha_{ii} \alpha_{ij}) = 0 \qquad (32)$$

This relationship (Equation (32)) is referred to as the orthogonality condition for coupled modes.

Higher Modes and Frequencies

Higher modes and frequencies beyond the fundamental can be obtained by setting up an auxiliary matrix, in a manner similar to that used for the uncoupled modes.

Express (14) as $\sum_{0}^{n} \bar{m}_i x_{ii} + \sum_{0}^{n} \bar{S}_i \alpha_{ii} = 0$ (33)

Express (15) as $\sum_{0}^{n} \bar{S}_i x_{ii} + \sum_{0}^{n} \bar{I}_i \alpha_{ii} = 0$ (34)

Multiply (33) by x_{oj} and (34) by α_{oj} and add the resulting equations. Then: $(\bar{M}_o x_{oj} + \bar{S}_o \alpha_{oj}) x_{oi} +$

$$(\bar{M}_1 x_{oj} + \bar{S}_1 \alpha_{oj}) x_{1i} + \cdots + (\bar{M}_n x_{oj} + \bar{S}_n \alpha_{oj}) x_{ni} + (\bar{S}_o x_{oj} + \bar{I}_o \alpha_{oj}) \alpha_{oi}$$
$$(\bar{S}_n x_{oj} + \bar{I}_n \alpha_{oj}) \alpha_{ni} = 0 \qquad (35)$$

Now (32) can be written in the form.

$$(m_0 x_{oj} + S_0\alpha_{oj})x_{oi} + (m_1 x_{1j} + \bar{S}_1\alpha_{1j})x_{1i} \cdots + (\bar{m}_n x_{nj} + S_n\alpha_{nj})x_{ni}$$

$$+ (\bar{S}_0 x_{oj} + I_0\alpha_{oj})\alpha_{oi} \cdots + (\bar{S}_n x_{nj} + I_n\alpha_{nj})\alpha_{ni} = 0 \quad (36)$$

Then subtracting (35) from (36) x_{oi} and α_{oi} are eliminated.

Then:

$$[\bar{m}_1(x_{1j} - x_{oj}) + \bar{S}_1(\alpha_{1j} - \alpha_{oj})]x_{1i} + [\bar{m}_2(x_{2j} - x_{oj}) + \bar{S}_2(\alpha_{2j} - \alpha_{oj})]x_{2i} +$$

$$\cdots [\bar{m}_n(x_{nj} - x_{oj}) + \bar{S}_n(\alpha_{nj} - \alpha_{oj})]x_{ni} + [\bar{S}_1(x_{1j} - x_{oj}) + \bar{I}_1(\alpha_{1j} - \alpha_{oj})]\alpha_{1i}$$

$$+ \cdots + [\bar{S}_n(x_{nj} - x_{oj}) + \bar{I}_n(\alpha_{nj} - \alpha_{oj})]\alpha_{ni} = 0 \quad (37)$$

or (38)

$$A_1 x_{1i} + A_2 x_{2i} + \cdots A_n x_{ni} + B_1\alpha_{1i} + B_2\alpha_{2i} + \cdots B_n\alpha_{ni} = 0$$

Now if the n^{th} element is used for normalizing then:

$$a_{n1} = \frac{A_1}{A_n}, \quad a_{n2} = \frac{A_2}{A_n}, \quad a_{nn} = 1 \cdots a_{n(n+1)} = \frac{B_1}{A_n}, \quad a_{n(2n)} = \frac{B_n}{A_n}$$

and [E] is of the form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \\ a_{n1} & a_{n2} & & 1 & a_{n(n+1)} \cdots & a_{n(2n)} \\ 0 & 0 & 0 & \cdots & & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & & 0 \end{bmatrix} \quad (39)$$

APPENDIX IV

THREE DEGREE - THREE DIMENSIONAL
FLUTTER THEORY

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Consider the motion of an airfoil section as the deflection downward of the elastic axis, a rotation about the elastic axis, and a rotation of the aileron about the aileron hinge line.

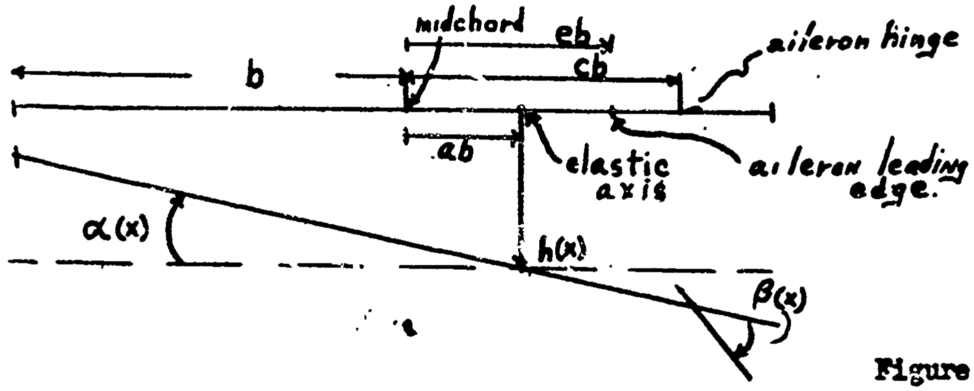

Figure 1

If $m(x)$ mass per unit span of wing-aileron, at station x (slugs)
$I_\alpha(x)$ mass moment of inertia per unit span, at station x, about the elastic axis, of wing-aileron (slug-ft^2)
$S_\alpha(x)$ static mass moment per unit span, at x of wing aileron about the elastic axis (slug-ft)
$I_\beta(x)$ aileron mass moment of inertia per unit span, at x, about the aileron hinge line (slug-ft^2)
$S_\beta(x)$ aileron static mass moment per unit length, at x about the aileron hinge line (slug-ft)
b semichord length, at station x, (ft)
$(c-a)\,b$ distance between the elastic axis and the aileron hinge in feet.

If the wing is moving with velocities $\dot{h}(x)$, $\dot{\alpha}(x)$, $\dot{\beta}(x)$, the kinetic energy per unit length of span is:

$$dT = \tfrac{1}{2} m(x)\dot{h}(x)^2 + \tfrac{1}{2} I_\alpha(x)\dot{\alpha}(x)^2 + \tfrac{1}{2} I_\beta(x)\dot{\beta}(x)^2 + S_\beta(x)\dot{h}(x)\dot{\beta}(x) + S_\alpha(x)\dot{h}(x)\dot{\alpha}(x) + \left[S_\beta(x)(c-a)b + I_\beta(x)\right]\dot{\beta}(x)\dot{\alpha}(x) \quad (1)$$

Equation (1) for the kinetic energy is the expression for a two dimensional system.

In order to simplify the analysis a basic assumption is made at this point. Namely, that the aerodynamic forces and moments do not change the shapes of the normal modes of vibration of the wing. Thus while the flutter frequency may vary, the modal shape is assumed to remain unchanged from the ground vibration normal modes.

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The motion of the wing can then be considered as a superposition of the normal modes.

If $f(x)$ denotes the uncoupled bending deflection mode, $f(\ell) = 1$ where ℓ is the semi-span of the wing.

$F(x)$ denotes the uncoupled torsion deflection mode $F(\ell) = 1$

$\varphi(x)$ denotes the aileron deflection mode $\varphi(\ell) = 1$

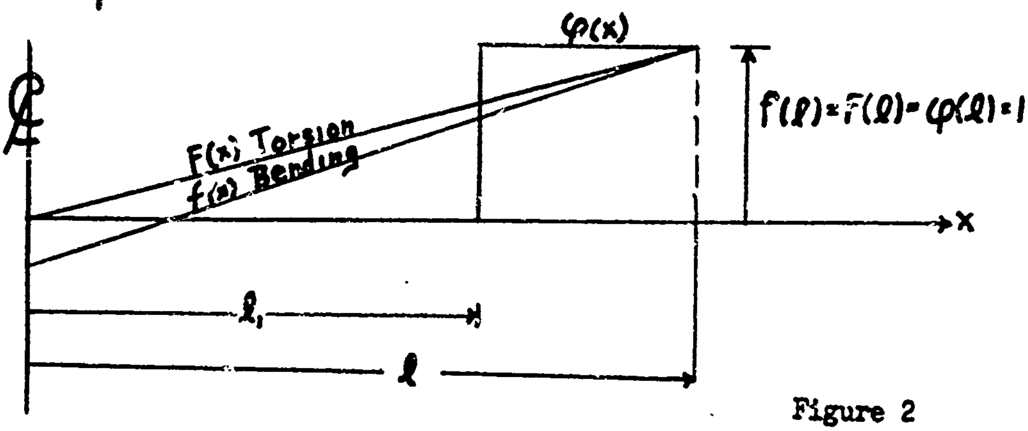

Figure 2

Then $h(x)$ can be replaced by $hf(x)$
$\alpha(x)$ can be replaced by $\alpha F(x)$
$\beta(x)$ can be replaced by $\beta\varphi(x)$

Now the aileron is usually very stiff structurally compared with the control system. If the deflection is therefore considered to be control system deflection only (a reasonable assumption), the deflection $\beta(x)$ along the span can be considered constant, i.e. $\varphi(x) = 1$.

The total kinetic energy then is:

$$T = \tfrac{1}{2}\dot{h}^2\int_0^{\ell} m(x)[f(x)]^2 dx + \tfrac{1}{2}\dot{\alpha}^2\int_0^{\ell} I_\alpha(x)[F(x)]^2 dx + \tfrac{1}{2}\dot{\beta}^2\int_{\ell_1}^{\ell_2} I_\beta(x)\, dx + \dot{h}\dot{\beta}\int_{\ell_1}^{\ell_2} S_\beta(x) f(x)\, dx + \dot{h}\dot{\alpha}\int_0^{\ell} S_\alpha(x) f(x) F(x)\, dx + \dot{\alpha}\dot{\beta}\int_{\ell_1}^{\ell_2}[S_\beta(x)(c-a)b + I_\beta(x)]F(x)\, dx \qquad (2)$$

(where $\int_0^{\ell}$ represents integration over the semi-wing and $\int_{\ell_1}^{\ell_2}$ represents integration over the aileron)

or

$$T = \tfrac{1}{2}M\dot{h}^2 + \tfrac{1}{2}I_\alpha\dot{\alpha}^2 + \tfrac{1}{2}\dot{\beta}^2 I_\beta + S_\beta\dot{h}\dot{\beta} + S_\alpha\dot{h}\dot{\alpha} + P_{\alpha\beta}\dot{\alpha}\dot{\beta} \qquad (3)$$

where in equation (3)

$$M = \int_0^l m(x)[f(x)]^2 dx \qquad I_\beta = \int_{l_1}^{l_2} I_\beta(x)\, dx$$

$$I_\alpha = \int_0^l I_\alpha(x)[F(x)]^2 dx \qquad S_\beta = \int_{l_1}^{l_2} S_\beta(x) f(x)\, dx$$

$$S_\alpha = \int_0^l S_\alpha(x) f(x) F(x)\, dx \qquad P_{\alpha\beta} = \int_{l_1}^{l_2} [S_\beta(x)(c-a)b + I_\beta(x)] F(x) dx$$

The above expressions for M I_α etc. can be considered physically as "weighted" mass of mechanical terms.

Potential Energy

The potential energy stored in the system when the system is deflected, is strain energy of bending, torsion and control system strain. The total strain energy can be written as:

$$U = \frac{1}{2}\int_0^l EI(x)\left\{\frac{\delta^2}{\delta x^2}[h f(x)]\right\}^2 dx + \frac{1}{2}\int_0^l GJ(x)\left\{\frac{\delta}{\delta x}[\alpha F(x)]\right\}^2 dx + \frac{1}{2}K_\beta \beta^2 \qquad (4)$$

where $EI(x)$ is bending rigidity of wing at station x.

$GJ(x)$ is torsional rigidity of wing at station x, (including interaction of spars and torque boxes).

k_β is torsional spring constant of aileron control system.

Then if $k_h = \int_0^l EI(x)\left\{\frac{\delta^2}{\delta x^2}[f(x)]\right\}^2 dx$

$k_\alpha = \int_0^l GJ(x)\left\{\frac{\delta}{\delta x}[F(x)]\right\}^2 dx$

the potential energy can then be expressed as:

$$U = \frac{1}{2}K_h h^2 + \frac{1}{2}K_\alpha \alpha^2 + \frac{1}{2}K_\beta \beta^2 \qquad (5)$$

The above expression (Equation (5)) for the potential energy contains quantities which are not easily calculable. In order to obtain simple expressions for the coefficients of the potential energy, suppose that the wing is vibrating in a vacumn, so restrained, that at any time, it is free to oscillate in simple harmonic motion, in but one degree of freedom; (i.e.)

(a) first restrain the wing so that it cannot twist, with aileron clamped to wing.
(b) then restrain wing against bending, aileron clamped, wing permitted to twist only,
(c) then restrain wing against bending and torsion, aileron free to oscillate.

For condition (a) $h = h_0 e^{i\omega_h t}$ $\qquad \ddot{h} = -\omega_h^2 h$

$\alpha = \beta = 0$

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(b) $\alpha = \alpha_0 e^{i\omega_\alpha t}$
$\ddot{\alpha} = -\omega_\alpha^2 \alpha$
$h = \beta = 0$

For (c) $\beta = \beta_0 e^{i\omega_\beta t}$
$\ddot{\beta} = -\omega_\beta^2 \beta$
$h = \alpha = 0$

For condition (a) Lagrange's Equation of Motion is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{h}}\right) + \frac{\partial U}{\partial h} = 0 = M\ddot{h} + k_h h \qquad (6)$$

or:

$$(-M\omega_h^2 + k_h)\, h = 0 \qquad (7)$$

hence

$$k_h = M\omega_h^2 \qquad (8)$$

Similarly for conditions (b) and (c) above the values of k_α and k_β can be expressed in terms of the uncoupled frequencies:

$$k_\alpha = I_\alpha \omega_\alpha^2$$
$$k_\beta = I_\beta \omega_\beta^2 \qquad (9)$$

Finally, then

$$U = \tfrac{1}{2} M\omega_h^2 h^2 + \tfrac{1}{2} I_\alpha \omega_\alpha^2 \alpha^2 + \tfrac{1}{2} I_\beta \omega_\beta^2 \beta^2 \qquad (10)$$

<u>Aerodynamic Forces (Ref. AAF TR 4798)</u>

The total aerodynamic lift per unit span is:

$$L' = \pi\rho b^2\omega^2 L_h h(x) + \pi\rho b^3\omega^2\left\{\left[L_\alpha - \left(\tfrac{1}{2}+a\right)L_h\right]\alpha(x) + \left[L_\beta - (c-e)L_z\right]\beta(x)\right\} \qquad (11)$$

The total oscillatory aerodynamic moment about the elastic axis per unit span is:

$$\bar{M}' = \pi\rho b^3\omega^2\left[M_h - \left(\tfrac{1}{2}+a\right)L_h\right]h(x) + \pi\rho b^4\omega^2\left\{\left[M_\alpha - \left(\tfrac{1}{2}+a\right)L_\alpha - \left(\tfrac{1}{2}+a\right)M_h + \left(\tfrac{1}{2}+a\right)^2 L_h\right]\alpha(x) + \left[M_\beta - \left(\tfrac{1}{2}+a\right)L_\beta - (c-e)M_z + (c-e)\left(\tfrac{1}{2}+a\right)L_z\right]\beta(x)\right\} \qquad (12)$$

The total oscillatory torque acting on the aileron about the aileron hinge line, per unit span is:

$$\bar{\bar{T}}' = \pi\rho b^3\omega^2\left[T_h - (c-e)P_h\right]h(x) + \pi\rho b^4\omega^2\left\{\left[T_\alpha - (c-e)P_\alpha - \left(\tfrac{1}{2}+a\right)T_h + \left(\tfrac{1}{2}+a\right)(c-e)P_h\right]\alpha(x) + \left[T_\beta - (c-e)(P_\beta + T_z) + (c-e)^2 P_z\right]\beta(x)\right\} \qquad (13)$$

Where in the above expressions the coefficients are defined in terms of Theodorson's T, F and G functions (NACA TR 496) and the aileron terms are for the aerodynamically balanced aileron and are defined by Kussner's φ functions (NACA TM 991). These coefficients can be summarized as:

$$L_h = 1 - 2j\left(\frac{v}{b\omega}\right)(F + jG)$$

$$L_\alpha = \frac{1}{2} - j\left(\frac{v}{b\omega}\right)[1 + 2(F + jG)] - 2\left(\frac{v}{b\omega}\right)^2(F + jG)$$

$$L_\beta = \frac{T_1}{\pi} + j\left(\frac{v}{b\omega}\right)\left(\frac{T_4}{\pi}\right) - j\left(\frac{v}{b\omega}\right)\frac{T_{11}}{\pi}(F + jG) - 2\left(\frac{v}{b\omega}\right)^2\frac{T_{10}}{\pi}(F + jG)$$

$$L_z = 1 - 2j\left(\frac{v}{b\omega}\right)\frac{\varphi_1}{\pi}(F + jG) + \frac{\varphi_2}{\pi}$$

$$M_h = \frac{1}{2}$$

$$M_\alpha = \frac{3}{8} - j\left(\frac{v}{b\omega}\right)$$

$$M_\beta = -\frac{T_7}{\pi} - \left(e + \frac{1}{2}\right)\frac{T_1}{\pi} + j\left(\frac{v}{b\omega}\right)\left(\frac{2\phi + T_4}{\pi}\right) - \left(\frac{v}{b\omega}\right)^2\left(\frac{T_4 + T_{10}}{\pi}\right)$$

$$M_z = -j\left(\frac{v}{b\omega}\right)\frac{\varphi_5}{\pi} + \frac{1}{4}\frac{\varphi_6}{\pi}$$

$$T_h = -\frac{T_1}{\pi} - j\left(\frac{v}{b\omega}\right)\frac{T_{12}}{\pi}(F + jG)$$

$$T_\alpha = -\frac{1}{\pi}\left[T_7 + \left(e + \frac{1}{2}\right)T_1\right] - j\left(\frac{v}{b\omega}\right)\left(\frac{2\phi - 2T_1 - T_4}{2\pi}\right) - j\left(\frac{v}{b\omega}\right)\frac{T_{12}}{\pi}(F + jG)$$
$$- \left(\frac{v}{b\omega}\right)^2\frac{T_{12}}{\pi}(F + jG)$$

$$T_\beta = -\left(\frac{T_3}{\pi^2}\right) + j\left(\frac{v}{b\omega}\right)\frac{T_4 T_{11}}{2\pi^2} - j\left(\frac{v}{b\omega}\right)\frac{T_{11} T_{12}}{2\pi^2}(F + jG)$$
$$- \left(\frac{v}{b\omega}\right)^2\left(\frac{T_5 - T_4 T_{10}}{\pi^2}\right) - \left(\frac{v}{b\omega}\right)^2\left(\frac{T_{10} T_{12}}{\pi^2}\right)(F + jG)$$

$$T_z = -j\left(\frac{v}{b\omega}\right)\frac{\varphi_1\varphi_2}{\pi^2}(F+jG) - j\left(\frac{v}{b\omega}\right)\left(\frac{\varphi_{10}}{\pi^2}\right)$$
$$+\frac{1}{2}\frac{\varphi_{37}}{\pi^2}$$

$$P_h = -2j\left(\frac{v}{b\omega}\right)\frac{\varphi_{31}}{\pi}(F+jG) + \frac{\varphi_3}{\pi}$$

$$P_\alpha = -2\left[\left(\frac{v}{b\omega}\right)^2 + j\left(\frac{v}{b\omega}\right)\right]\frac{\varphi_{31}}{\pi}(F+jG) - j\left(\frac{v}{b\omega}\right)\frac{\varphi_{32}}{\pi}$$
$$+\frac{1}{4}\frac{\varphi_6}{\pi}$$

$$P_\beta = -\frac{2}{\pi}\left[\left(\frac{v}{b\omega}\right)^2\varphi_1 + \frac{1}{2}j\left(\frac{v}{b\omega}\right)\varphi_2\right]\frac{\varphi_{31}}{\pi}(F+jG) - \left(\frac{v}{b\omega}\right)^2\frac{\varphi_{35}}{\pi^2}$$
$$-j\left(\frac{v}{b\omega}\right)\frac{\varphi_{36}}{\pi^2} + \frac{1}{2}\frac{\varphi_{37}}{\pi^2}$$

$$P_z = -2j\left(\frac{v}{b\omega}\right)\frac{\varphi_1\varphi_{31}}{\pi^2}(F+jG) - j\left(\frac{v}{b\omega}\right)\frac{\varphi_{35}}{\pi^2}$$
$$+\frac{\varphi_{17}}{\pi^2}$$

Generalized Forces

The generalised force in the h degree of freedom Q_h is determined from the virtual work done by displacing the structure from h to $\delta h + h$ (by the air forces), all other degrees of freedom being held constant during the displacement. Then

$$\delta W = L' \delta h(x) = L' f(x) \delta h$$

$$Q'_h = \frac{\delta W}{\delta h} = L' f(x)$$

Similarly

$$Q'_\alpha = \frac{\delta W}{\delta \alpha} = \bar{M}' F(x)$$

$$Q_\beta = \frac{\delta W}{\delta \beta} = \bar{T}' \varphi(x) = \bar{T}' \quad (\text{SINCE } \varphi(x) = 1)$$

And for the entire wing:

$$Q_h = \int_0^l L' f(x)\, dx$$

$$Q_\alpha = \int_0^l \bar{M}' F(x)\, dx \qquad (14)$$

$$Q_\beta = \int_{l_1}^l \bar{T}'\, dx$$

If now in equations (11) (12) and (13) $h(x)$ is replaced by $hf(x)$, $\alpha(x)$ by $\alpha F(x)$ and $\beta(x)$ by β then the equations for Q_h, Q_α, Q_β (equation (14)) can be written as:

$$Q_h = \pi \rho \omega^2 \left[A_{hh} h + A_{h\alpha} \alpha + A_{h\beta} \beta \right]$$

$$Q_\alpha = \pi \rho \omega^2 \left[A_{\alpha h} h + A_{\alpha\alpha} \alpha + A_{\alpha\beta} \beta \right]$$

$$Q_\beta = \pi \rho \omega^2 \left[A_{\beta h} h + A_{\beta\alpha} \alpha + A_{\beta\beta} \beta \right]$$

Where

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$$A_{hh} = \int_0^l b^2 [f(x)]^2 \, dx \; L_h$$

$$A_{h\alpha} = \int_0^l b^3 f(x) F(x) \left[L_\alpha - (\tfrac{1}{2} + a) L_h \right] dx$$

$$A_{h\beta} = \int_{l_1}^l b^3 f(x) \left[L_\beta - (c-e) L_z \right] dx$$

$$A_{\alpha h} = \int_0^l b^3 f(x) F(x) \left[M_h - (\tfrac{1}{2} + a) L_h \right] dx$$

$$A_{\alpha\alpha} = \int_0^l b^4 [F(x)]^2 \left[M_\alpha - (\tfrac{1}{2}+a)(L_\alpha + M_h) + (\tfrac{1}{2}+a)^2 L_h \right] dx$$

$$A_{\alpha\beta} = \int_{l_1}^l b^4 F(x) \left[M_\beta - (\tfrac{1}{2} + a) L_\beta - (c-e) M_z + (c-e)(\tfrac{1}{2}+a) L_z \right] dx$$

$$A_{\beta h} = \int_{l_1}^l b^3 f(x) \left[T_h - (c-e) P_h \right] dx$$

$$A_{\beta\alpha} = \int_{l_1}^l b^4 F(x) \left[T_\alpha - (c-e) P_\alpha - (\tfrac{1}{2}+a) T_h + (\tfrac{1}{2}+a)(c-e) P_h \right] dx$$

$$A_{\beta\beta} = \int_{l_1}^l b^4 \left[T_\beta - (c-e)(P_\beta + T_z) + (c-e)^2 P_z \right] dx$$

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It is to be noted that the evaluation of L_h .... P_z becomes more difficult for the tapered, three dimensional wing that for the two dimensional case. For every assumed value of $1/k$, $\frac{v}{b\omega}$ varies along the span. The aerodynamic coefficients must therefore be included under the integral sign. Needless to say the computation of the aerodynamic terms A_{hh} .... $A_{\beta\beta}$ becomes long and tedious

Now, since $(F + jG)$ does not vary rapidly with $\frac{v}{b\omega}$, it will be assumed that $F + jG$ is constant along the span for any assumed value of $1/k$. If b_r is the semi chord at the 3/4 span position then $(F + jG)$ is assumed to be a function of $\frac{v}{b_r\omega} = \frac{1}{k}$.

The aerodynamic terms are functions of $\frac{v}{b\omega}$ explicitly as well as thru $F + jG$. To simplify the analysis further assume that, for the aileron span the term $\frac{v}{b\omega}$ can be replaced by $\frac{v}{b_r\omega}$, (the 3/4 span position is usually approximately at the 50% aileron position.) Therefore, the aileron aerodynamic terms can be expressed as functions of $\frac{v}{b_r\omega}$ and e.

However, for the four wing bending-torsion aerodynamic terms it is incorrect to replace $\frac{v}{b\omega}$ by $\frac{v}{b_r\omega}$.

Examination of L_h, L_α, and M_α on page 6 shows that these terms are functions of $\frac{v}{b\omega}$, and $\left(\frac{v}{(b\omega)}\right)^2$. They can be rewritten as:

$$L_h = 1 + \frac{b_r}{b}\left[-2j\left(\frac{v}{b_r\omega}\right)(F + jG)\right.$$

$$L_\alpha = \frac{1}{2} + \frac{b_r}{b}\left[-j\left(\frac{v}{b_r\omega}\right) - 2j\left(\frac{v}{b_r\omega}\right)(F + jG)\right] + \left(\frac{b_r}{b}\right)^2\left[-2\left(\frac{v}{b_r\omega}\right)(F + jG)\right]$$

$$M_\alpha = \frac{3}{8} + \frac{b_r}{b}\left[-\left(\frac{v}{b_r\omega}\right)\right]$$

or

$$L_h = K_1(L_h) + \frac{b_r}{b} K_2(L_h)$$

$$L_\alpha = K_1(L_\alpha) + \frac{b_r}{b} K_2(L_\alpha) + \left(\frac{b_r}{b}\right)^2 K_3(L_\alpha)$$

$$M_\alpha = K_1(M_\alpha) + \frac{b_r}{b} K_2(M_\alpha)$$

Where $K_1 \ (L_h)$, (to be read as K_1 of function L_h) is the constant part of L_h, $K_2(L_h)$ is the part of function L_h which is a function of $\frac{v}{b_r \omega}$; $K_3(L_\alpha)$ is the part of function L_α which is function of $\left(\frac{v}{b_r \omega}\right)^2$, etc. Since $K_1(L_h) = 1$, $K_1(L_\alpha) = .5$, $K_1(M_\alpha) = .375$ and $M_h = .5$ the expressions for A_{hh}, $A_{h\alpha}$, $A_{\alpha h}$ and $A_{\alpha\alpha}$ finally become:

$$A_{hh} = \int_0^l b^2 [f(x)]^2 dx + b_r K_2 L_h \int_0^l b [f(x)]^2 dx$$

$$A_{h\alpha} = -\int_0^l a b^3 f(x) F(x) dx + b_r K_2(L_\alpha) \int_0^l b^2 f(x) F(x) dx$$
$$+ b_r^2 K_3(L_\alpha) \int_0^l b f(x) F(x) dx - b_r K_2(L_h) \int_0^l \left(\tfrac{1}{2} + a\right) b^2 f(x) F(x) dx$$

$$A_{\alpha h} = -\int_0^l a b^3 f(x) F(x) dx - b_r K_2(L_h) \int_0^l \left(\tfrac{1}{2} + a\right) b^2 f(x) F(x) dx$$

$$A_{\alpha\alpha} = \int_0^l \left(\tfrac{1}{8} + a^2\right) b^4 [F(x)]^2 dx + b_r K_2(M_\alpha) \int_0^l b^3 [F(x)]^2 dx$$
$$+ b_r K_2(L_h) \int_0^l \left(\tfrac{1}{2} + a\right)^2 b^3 [F(x)]^2 dx - b_r K_2(L_\alpha) \int_0^l \left(\tfrac{1}{2} + a\right) b^3 [F(x)]^2 dx$$
$$- b_r^2 K_3(L_\alpha) \int_0^l \left(\tfrac{1}{2} + a\right) b^2 [F(x)]^2 dx$$

Equations of Motion

Lagrange's equation in i^{th} degree of freedom is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = Q_i$$

where there is one equation for each degree of freedom. (16)

Then:

$$M\ddot{h} + S_\alpha\ddot{\alpha} + S_\beta\ddot{\beta} + M\omega_h^2 h = \pi\rho\omega^2\{A_{hh}h + A_{h\alpha}\alpha + A_{h\beta}\beta\}$$

$$S_\alpha\ddot{h} + I_\alpha\ddot{\alpha} + P_{\alpha\beta}\ddot{\beta} + I_\alpha\omega_\alpha^2\alpha = \pi\rho\omega^2\{A_{\alpha h}h + A_{\alpha\alpha}\alpha + A_{\alpha\beta}\beta\}$$

$$S_\beta\ddot{h} + P_{\alpha\beta}\ddot{\alpha} + I_\beta\ddot{\beta} + I_\beta\omega_\beta^2\beta = \pi\rho\omega^2\{A_{\beta h}h + A_{\beta\alpha}\alpha + A_{\beta\beta}\beta\}$$

By definition at the flutter speed the system vibration in simple harmonic motion with frequency ω.

$$\therefore \quad \ddot{h} = -\omega^2 h$$

$$\ddot{\alpha} = -\omega^2\alpha$$

$$\ddot{\beta} = -\omega^2\beta \qquad (17)$$

If (17) be substituted into equation (16) the equations of motion represent the motion of a system, with zero structural damping. If now structural damping is considered the equations must be modified. It has been found that the structural damping g is a function of the amplitude and not of frequency. Damping can be described by a force of magnitude proportional to the elastic restoring force, and 90° out of phase. Each restoring force term in (16) must then be modified by changing:

from $M_h\omega_h^2 h$ to $(1 + jg_h)M\omega_h^2 h$.

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from $I_\alpha \omega_\alpha^2 \alpha$ to $(1+jg_\alpha) I_\alpha \omega_\alpha^2 \alpha$ (18)

from $I_\beta \omega_\beta^2 \beta$ to $(1+jg_\beta) I_\beta \omega_\beta^2 \beta$

Making the indicated substitutions (17) and (18) in the dynamic equations (16) and grouping terms:

(19)

$$\left[M + \pi\rho A_{hh} - M(1+jg_h)\left(\frac{\omega_h}{\omega}\right)^2\right]h + \left[S_\alpha + \pi\rho A_{h\alpha}\right]\alpha + \left[S_\beta + \pi\rho A_{h\beta}\right]\beta = 0$$

$$\left[S_\alpha + \pi\rho A_{\alpha h}\right]h + \left[I_\alpha + \pi\rho A_{\alpha\alpha} - I_\alpha(1+ig_\alpha)\left(\frac{\omega_\alpha}{\omega}\right)^2\right]\alpha + \left[P_{\alpha\beta} + \pi\rho A_{\alpha\beta}\right]\beta = 0$$

$$\left[S_\beta + \pi\rho A_{\beta h}\right]h + \left[P_{\alpha\beta} + \pi\rho A_{\beta\alpha}\right]\alpha + \left[I_\beta + \pi\rho A_{\beta\beta} - I_\beta(1+ig_\beta)\left(\frac{\omega_\beta}{\omega}\right)^2\right]\beta = 0$$

For a solution of the above homogenous simultaneous equations to exist (other than the trivial case of $h = \alpha = \beta = 0$), the determinant of the coefficients must vanish.

A number of methods exist for the solution of the above determinantal equation. For any 1/k value only one unknown appears explicitly, (the flutter frequency ω). If $X = \left(\frac{\omega_\alpha}{\omega}\right)^2$ then the expansion of the determinant yields a complex cubic equation in X. For a solution to exist the real and imaginary parts must be zero separately. This condition of X satisfying the real and imaginary equations identically is satisified only for certain values of 1/k. Thus the solution involves the determination of two unknowns, 1/k and X. While the flutter frequency thru X is an unknown to be determined, the second unknown is not restricted to 1/k. Any of the physical parameters, such as elastic stiffness $\left(\frac{\omega_h}{\omega_\alpha}\right)^2$, damping coefficient g, or aileron balance, could be chosen for the second variable. Thus for any speed determined by the assumed value of 1/k, it is possible to determine the value of the second parameter which would permit the existence of flutter at that speed.

One of the most common methods employed in this country is to permit the damping factor g to be the second variable. If it is

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assumed that $g_h = g_\alpha = g_\beta = g$, $X = \left(\frac{\omega_\alpha}{\omega}\right)^2$, $Z = X(1+jg)$, $p = \left(\frac{\omega_h}{\omega_\alpha}\right)^2$

$n = \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2$ then the flutter determinant can be expressed as:

$$\begin{vmatrix} \bar{A}_{hh} - pZ & \bar{A}_{h\alpha} & \bar{A}_{h\beta} \\ \bar{A}_{\alpha h} & \bar{A}_{\alpha\alpha} - Z & \bar{A}_{\alpha\beta} \\ \bar{A}_{\beta h} & \bar{A}_{\beta\alpha} & \bar{A}_{\beta\beta} - nZ \end{vmatrix} = 0 \qquad (20)$$

Where

$$\bar{A}_{hh} = 1 + \frac{\pi\rho A_{hh}}{M}; \quad \bar{A}_{h\alpha} = \frac{S_\alpha + \pi\rho A_{h\alpha}}{M}; \quad \bar{A}_{h\beta} = \frac{S_\beta + \pi\rho A_{h\beta}}{M}$$

$$\bar{A}_{\alpha h} = \frac{S_\alpha + \pi\rho A_{\alpha h}}{I_\alpha}; \quad \bar{A}_{\alpha\alpha} = 1 + \frac{\pi\rho A_{\alpha\alpha}}{I_\alpha}; \quad \bar{A}_{\alpha\beta} = \frac{P_{\alpha\beta} + \pi\rho A_{\alpha\beta}}{I_\alpha}$$

$$\bar{A}_{\beta h} = \frac{S_\beta + \pi\rho A_{\beta h}}{I_\beta}; \quad \bar{A}_{\beta\alpha} = \frac{P_{\alpha\beta} + \pi\rho A_{\beta\alpha}}{I_\beta}; \quad \bar{A}_{\beta\beta} = 1 + \frac{\pi\rho A_{\beta\beta}}{I_\beta}$$

Expansion of the determinant set equal to zero results in a complex cubic equation in Z for each 1/k. The roots of the complex equation can be expressed as:

$$Z = X + jY$$

then $g = \frac{Y}{X}$ and $\frac{V}{\omega_\alpha} = b_r \frac{1/k}{\sqrt{X}}$

or $$\frac{V \text{ mph}}{\omega_\alpha \text{ cpm}} = \frac{b_r}{14} \frac{1/k}{\sqrt{X}}$$

For each value of 1/k there are three values of g and $\frac{V}{\omega_\alpha}$. If at any value of 1/k the most critical root only is considered (root giving

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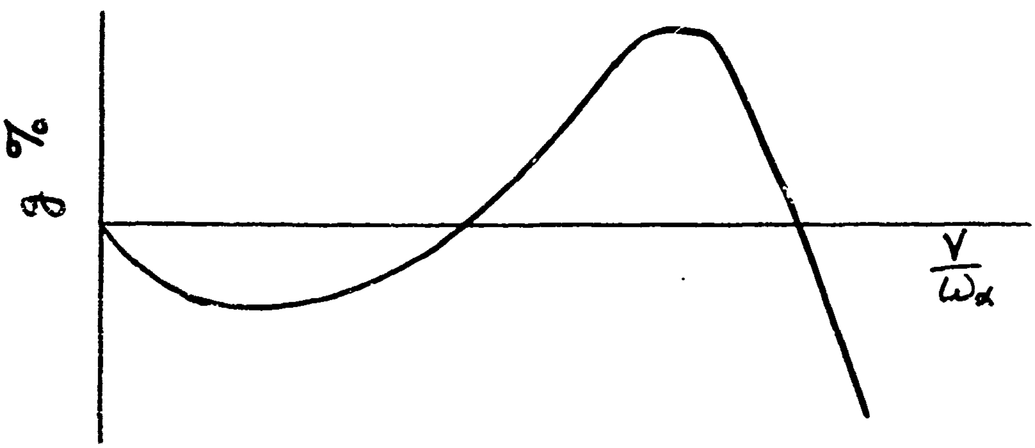
lowest negative value of g or highest positive value) then a typical g-V curve results.



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APPENDIX V

WING FLUTTER ANALYSIS BASED
ON
COUPLED VIBRATION MODES

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In a natural vibration mode of an airplane wing about equilibrium the whole system is (by definition) vibrating sinusoidally in such a way that each particle is in (or exactly out of) phase with every other particle. Hence any natural mode of vibration of the wing can be described basically by only two displacement components, bending and torsion, in the vertical plane containing an assumed reference axis and in planes perpendicular to this axis. For this the usual assumption must be made that the wing is rigid in planes perpendicular to the assumed reference axis. Thus the i^{th} coupled mode may be defined as follows:

Bending component: $\xi_i(t)\ h_i(x)$

(1) Torsion component: $\xi_i(t)\ \alpha_i(x)$

Where ξ_i is sinusoidal in time.

(The reference axis need not be the elastic axis.)

Under a general vibratory motion having small displacements from equilibrium, the displacements at any point along the reference axis will consist of superposed contributions of the various natural modes:

$$h(x,t) = \sum^{\infty} a_i \xi_i(t) h_i(x) = \sum^{\infty} \xi_i(t) h_i(x)$$

$$\alpha(x,t) = \sum^{\infty} a_i \xi_i(t) h_i(x) = \sum^{\infty} \xi_i(t) \alpha_i(x)$$

It may be noted that in h and α the coefficients a_i (which are immediately absorbed into the convenient generalized coordinates express "how much" of each normal mode ξ_i is introduced. The a_i are necessarily the same for corresponding contributions to bending and to torsion. This is due to the fact that bending and torsion in any natural mode are not independent but locked in phase relation and interrelated by the characteristics of that mode.

In practice it is considered that modes higher than the third seldom contribute much to the total displacement in any arbitrary motion. Hence only the first three natural modes will be employed in the subsequent analysis. The generalization to a higher number of modes is immediate.

Further, we shall assume that the wing is not continuous but composed of a finite set of chordwise strips numbered 1 to n. Thus $h(x,t)$ and $\alpha(x,t)$ throughout the wing will be replaced by

(3) $$h_k = h_{1k}\xi_1 + h_{2k}\xi_2 + h_{3k}\xi_3$$

(4) $$\alpha_k = \alpha_{1k}\xi_1 + h_{2k}\xi_2 + h_{3k}\xi_3$$

at each strip k.

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Case A - Motion Expressed in Terms of Generalized Coordinates

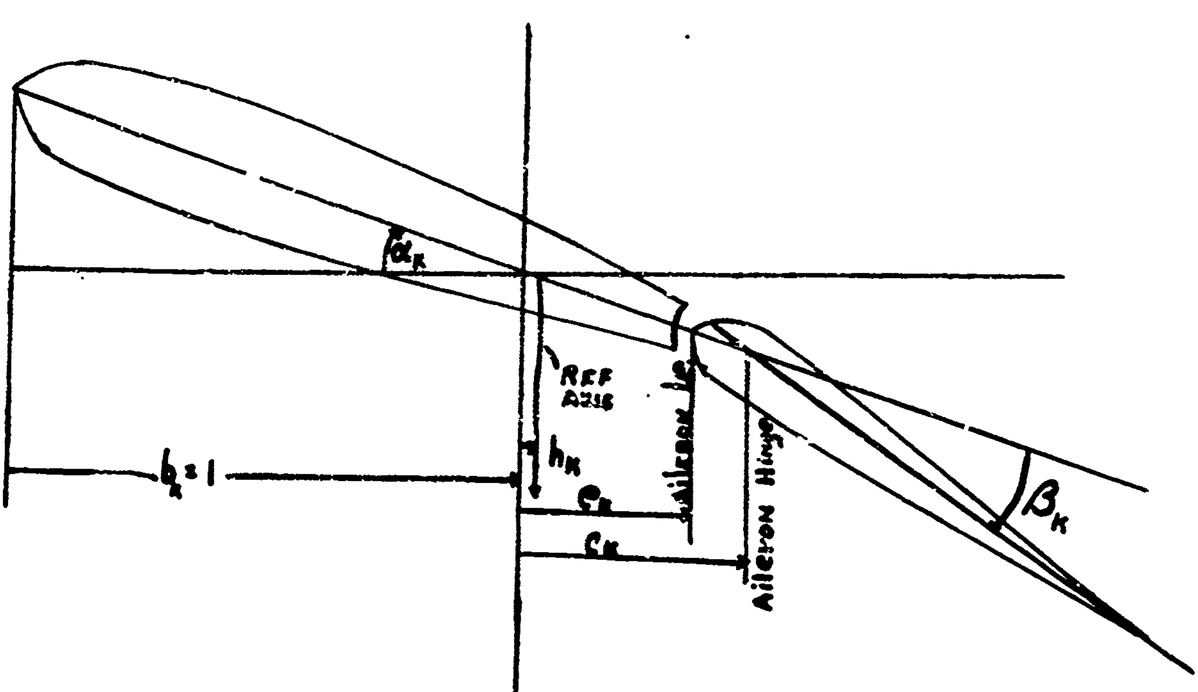
Usually the aileron degree of freedom β is not measured during ground vibration test nor is it calculated during coupled mode analysis. However, since it is an element vibrating sinusoidally in phase with the rest of the system during (undamped) normal modes, these modes can be measured and described so as to include the aileron also. We then have simply during a general motion

(5) $$\beta_k = \beta_{1k}\,\xi_1 + \beta_{2k}\,\xi_2 + \beta_{3k}\,\xi_3$$

(6) The kinetic energy of a single strip k of wing is given by

$$T_k = \tfrac{1}{2}\int_{-b_k}^{b_k e_k} \mu\left(\dot h_k + [z - a_k]\dot\alpha_k\right)^2 dz + \tfrac{1}{2}\int_{b_k e_k}^{b_k} \mu\left[\dot h_k + (z - a_k)\dot\alpha_k + (z - c_k)\dot\beta_k\right]^2 dz$$

Where $\left\{\begin{array}{l}\mu = \text{wing strip density}\\ z = \text{chordwise coordinate}\\ a, b, c, e\end{array}\right\}$ are given in the diagram below



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Measure all quantities in terms of the unit b and positive if aft of the midchord.

Thus

$$T_K = \frac{1}{2}\int_{-b_K}^{b_K e_K} \mu\left[\dot h_K^2 + 2(z-a_K)\dot h_K \dot\alpha_K + (z-a_K)^2\dot\alpha_K^2\right]dz +$$

$$⑦ \quad + \frac{1}{2}\int_{b_K e_K}^{b_K} \mu\left[\dot h_K^2 + (z-a_K)^2\dot\alpha_K^2 + (z-c_K)^2\dot\beta_K^2 + 2(z-a_K)\dot h_K\dot\alpha_K\right.$$

$$\left. + 2(z-c_K)\dot h_K\dot\beta_K + 2(z-a_K)(z-c_K)\dot\alpha\dot\beta\right]dz$$

or

$$T_K = \dot h_K^2\int_{-b_K}^{b_K}\frac{\mu}{2}\,dz + \dot\alpha_K^2\int_{-b_K}^{b_K}\frac{\mu}{2}(z-a_K)^2dz + \dot h_K\dot\alpha_K\int_{-b_K}^{b_K}\mu(z-a_K)dz$$

$$⑧ \quad + \dot\beta_K^2\int_{b_K e_K}^{b_K}\frac{\mu}{2}(z-c_K)^2dz + \dot h_K\dot\beta_K\int_{b_K e_K}^{b_K}\mu(z-c_K)\,dz$$

$$+ \dot\alpha_K\dot\beta_K\int_{b_K e_K}^{b_K}\mu(z-a_K)(z-c_K)\,dz$$

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The last term may be written

$$\dot{\alpha}_k \beta_k \int_{b_k e_k}^{b_k} \mu [(Z - C_k) + (C_k - a_k)](Z - C_k)\, dZ$$

(9)
$$= \dot{\alpha}_k \dot{\beta}_k \left\{ \int_{b_k e_k}^{b_k} \mu (Z - C_k)^2 dZ + \int_{b_k e_k}^{b_k} \mu (C_k - a_k)(Z - C_k)\, dZ \right\}$$

The expression for T_k can then be written

(10)
$$T_k = \tfrac{1}{2} M_k \dot{h}_k^2 + \tfrac{1}{2} I_{\alpha k} \dot{\alpha}_k^2 + \tfrac{1}{2} I_{\beta k} \dot{\beta}^2 + S_{\alpha k} \dot{\alpha}_k \dot{h}_k + S_{\beta k} \dot{h}_k \dot{\beta}_k + \left[S_{\beta k}(C_k - a_k) b_k + I_{\beta k} \right] \dot{\alpha}_k \dot{\beta}_k$$

where the notation has obvious definition.

(The total kinetic energy of the wing is $\sum_k T_k$)

The expressions M_k, $I_{\alpha k}$ etc., can be interpreted as follows for strip k:

M_k = mass of wing-aileron combination

$I_{\alpha k}$ = mass moment of inertia of wing-aileron combination about reference axis.

$S_{\alpha k}$ = static moment of wing-aileron combination about reference axis.

$I_{\beta k}$ = mass moment of inertia of aileron about its hinge line.

$S_{\beta k}$ = static moment of aileron about its hinge line.

We now substitute the expressions (3), (4), and (5) into (10) and sum on k to get the total kinetic energy.

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$$T = \sum_k \Big\{ \tfrac{1}{2} M_k (h_{1k}\dot{\xi}_1 + h_{2k}\dot{\xi}_2 + h_{3k}\dot{\xi}_3)^2 + \tfrac{1}{2} I_{\alpha k}(\alpha_{1k}\dot{\xi}_1 + \alpha_{2k}\dot{\xi}_2 + \alpha_{3k}\dot{\xi}_3)^2 +$$

$$+ \tfrac{1}{2} I_{\beta k}(\beta_{1k}\dot{\xi}_1 + \beta_{2k}\dot{\xi}_2 + \beta_{3k}\dot{\xi}_3)^2 + S_{\alpha k}(h_{1k}\dot{\xi}_1 + h_{2k}\dot{\xi}_2 + h_{3k}\dot{\xi}_3)(\alpha_{1k}\dot{\xi}_1 + \alpha_{2k}\dot{\xi}_2 + \alpha_{3k}\dot{\xi}_3)$$

$$(11) \quad + S_{\beta k}(h_{1k}\dot{\xi}_1 + h_{2k}\dot{\xi}_2 + h_{3k}\dot{\xi}_3)(\beta_{1k}\dot{\xi}_1 + \beta_{2k}\dot{\xi}_2 + \beta_{3k}\dot{\xi}_3) +$$

$$+ [I_{\beta k} + S_{\beta k}(c_k - a_k)b_k](\alpha_{1k}\dot{\xi}_1 + \alpha_{2k}\dot{\xi}_2 + \alpha_{3k}\dot{\xi}_3)(\beta_{1k}\dot{\xi}_1 + \beta_{2k}\dot{\xi}_2 + \beta_{3k}\dot{\xi}_3) \Big\}$$

Now in the expression above, cross-product terms in the $\dot{\xi}_i$ must vanish since the $\dot{\xi}_i$ are originally chosen as proportional to normal coordinates. Setting each of the coefficients of cross-product terms equal to zero separately defines the generalized orthogonality condition of the normal (coupled) modes.

Thus

$$(12) \quad T = \tfrac{1}{2}(A_1\dot{\xi}_1^2 + A_2\dot{\xi}_2^2 + A_3\dot{\xi}_3^2)$$

where

$$A_i = \sum_k \Big\{ M_k h_{ik}^2 + I_{\alpha k}\alpha_{ik}^2 + I_{\beta k}\beta_k^2 + 2\big[S_{\alpha k} h_{ik}\alpha_{ik} +$$

$$(13) \quad + S_{\beta k} h_{ik}\beta_{ik} + (I_{\beta k} + S_{\beta k}[c_k - a_k]b_k)\alpha_{ik}\beta_{ik}\big]\Big\}$$

The generalized orthogonality conditions are:

(14)
$$\sum_k \{ M_k h_{ik} h_{jk} + I_{\alpha k} \alpha_{ik} \alpha_{jk} + S_{\alpha k}(h_{ik}\alpha_{jk} + \alpha_{ik} h_{jk}) + I_{\beta k} \beta_{ik}\beta_{jk} + S_{\beta k}(h_{ik}\beta_{jk} + \beta_{ik} h_{jk}) + [I_{\beta k} + S_{\beta k}(c_k - a_k) b_k](\alpha_{ik}\beta_{jk} + \beta_{ik}\alpha_{jk}) \} = 0$$
$$i = 1,2 \quad j = 2,3 \qquad i \neq j$$

It should be noted here that only if the aileron motion (as analytically described) is dynamically related to the wing motion (e.g., by direct test results) may the β degree of freedom be included as a normal coordinate. Otherwise the β degree of freedom must be carried as an extra coordinate and the results must be expressed in quasi-normal coordinates. (See Case B) The generalised orthogonality condition as written above holds only for strictly normal coordinates.

Next, the potential energy expression is developed, For a given strip k the potential energy U_k is expressible as a quadratic function of the displacements h, α, β. Thus it is also a quadratic function of the ξ_i. But this quadratic function must contain only squared terms since the ξ_i were originally chosen as proportional to normal coordinates. Thus

(15)
$$U = \sum_k U_k = \frac{1}{2}(B_1 \xi_1^2 + B_2 \xi_2^2 + B_3 \xi_3^2)$$

If the case of free oscillations without external forces is considered, the B_i may be evaluated. Using Lagrange's equations of motion:

(16)
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_i}\right) + \frac{\partial U}{\partial \xi_i} = 0$$

there are obtained

(17)
$$A_i \ddot{\xi}_i + B_i \xi_i = 0$$

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which are the equations for simple harmonic motion in the normal modes ξ_i. In these cases $\ddot{\xi}_i = -\omega_i^2 \xi_i$ where ω_i is the circular frequency of oscillation in the i^{th} normal mode. Hence

(18) $$A_i(-\omega_i^2)\xi_i + B_i \xi_i = 0$$

or

(19) $$B_i = \omega_i^2 A_i$$

Then the potential energy expression is

(20) $$U = \frac{1}{2}(\omega_1^2 A_1 \xi_1^2 + \omega_2^2 A_2 \xi_2^2 + \omega_3^2 A_3 \xi_3^2)$$

In the case where flutter occurs at the frequency ω the flutter forces and moments in the h, α and β degrees of freedom are given by AAF TR4798 1 as:

Lift at reference axis per unit span at strip k:

(21) $$L_k = \pi\rho b_k^3 \omega^2 \left\{ \frac{h_k}{b_k} L_h + \alpha_k\left[L_\alpha - L_h(\tfrac{1}{2}+a_k)\right] + \beta_k\left[L_\beta - L_z(c_k - e_k)\right]\right\}$$

Moment at reference axis per unit span at strip k:

(22) $$\bar{M}_k = \pi\rho b_k^4 \omega^2 \left\{ \frac{h_k}{b_k}\left[M_h - L_h(\tfrac{1}{2}+a_k)\right] + \alpha_k\left[M_\alpha - L_\alpha(\tfrac{1}{2}+a) - M_h(\tfrac{1}{2}+a_k) + L_h(\tfrac{1}{2}+a_k)^2\right] + \beta_k\left[M_\beta - L_\beta(\tfrac{1}{2}+a_k) - M_z(c_k - e_k) + L_z(c_k - e_k)(\tfrac{1}{2}+a_k)\right]\right\}$$

Moment about aileron hinge per unit span at strip k:

(23) $$\bar{M}_k = \pi\rho b_k^4 \omega^2 \left\{ \frac{h_k}{b_k}\left[T_h - P_h(c_k - e_k)\right] + \alpha_k\left[T_\alpha - P_\alpha(c_k - e_k) - T_h(\tfrac{1}{2}+a_k) + P_h(\tfrac{1}{2}+a_k)(c_k - e_k)\right] + \beta_k\left[T_\beta - (P_\beta + T_z)(c_k - e_k) + P_z(c_k - e_k)^2\right]\right\}$$

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above, the terms L_h, L_α, L_β, L_z, M_h, M_α, M_β, M_z, ., T_h, P_h, T_α, P_α, T_β, P_β, T_z, P_z, are available in tables for various 1/k values (or constants) and are listed in Ref. 1, Pages 32-34.

Now to develop the expressions for generalized forces corresponding to the generalized coordinates ξ_i we let ξ_i take on a virtual change of amount $\delta \xi_i$. The work done at strip k is

(24) $$\delta W_K = (L_K \delta h_K + \bar{M}_K \delta\alpha_K + \bar{\bar{M}}_K \delta\beta_K) \Delta x_K$$

where Δx_K is spanwise width of strip k,

$$\delta h_K = h_{iK} \delta \xi_i \qquad [= h_K(\xi_i + \delta \xi_i) - h_K(\xi_i)]$$

$$\delta\alpha_K = \alpha_{iK} \delta \xi_i$$

$$\delta\beta_K = \beta_{iK} \delta \xi_i$$

so that

(25) $$\delta W_K = (L_K h_{iK} + \bar{M}_K \alpha_{iK} + \bar{\bar{M}}_K \beta_{iK}) \delta \xi_i \Delta x_K$$

and thus

(26) $$Q_i = \sum_K \frac{\delta W_K}{\delta \xi_i} = \sum_K [L_K h_{iK} + \bar{M}_K \alpha_{iK} + \bar{\bar{M}}_K \beta_{iK}] \Delta x_K$$

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Thus

$$Q_i = \sum_k \Big\{ \pi\rho b_k^3 \omega^2 \Big[\frac{h_{1k}\xi_1 + h_{2k}\xi_2 + h_{3k}\xi_3}{b_k} L_h +$$

$$+ (\alpha_{1k}\xi_1 + \alpha_{2k}\xi_2 + \alpha_{3k}\xi_3)\big[L_\alpha - (\tfrac{1}{2}+a_k)L_h\big] +$$

$$+ (\beta_{1k}\xi_1 + \beta_{2k}\xi_2 + \beta_{3k}\xi_3)\big[L_\beta - L_z(c_k - e_k)\big]\Big]\Big\} h_{ik}\,\Delta x_k$$

$$+ \sum_k \Big\{ \pi\rho b_k^4 \omega^2 \Big[\frac{h_{1k}\xi_1 + h_{2k}\xi_2 + h_{3k}\xi_3}{b_k} \big(M_h - (\tfrac{1}{2}+a_k)L_h \big) +$$

(27)

$$+ (\alpha_{1k}\xi_1 + \alpha_{2k}\xi_2 + \alpha_{3k}\xi_3)\big[M_\alpha - (\tfrac{1}{2}+a_k)(L_\alpha + M_h) + (\tfrac{1}{2}+a_k)^2 L_h\big] +$$

$$+ (\beta_{1k}\xi_1 + \beta_{2k}\xi_2 + \beta_{3k}\xi_3)\big[M_\beta - (\tfrac{1}{2}+a_k)L_\beta - (c_k - e_k)M_z + (c_k - e_k)(\tfrac{1}{2}+a_k)L_z\big]\Big\} \alpha_{ik}\,\Delta x_k$$

$$+ \sum_k \Big\{ \pi\rho b^4 \omega^2 \Big[\frac{h_{1k}\xi_1 + h_{2k}\xi_2 + h_{3k}\xi_3}{b_k} \big(T_h - (c_k - e_k) P_h \big) +$$

$$+ (\alpha_{1k}\xi_1 + \alpha_{2k}\xi_2 + \alpha_{3k}\xi_3)\big[T_\alpha - (c_k - e_k)P_\alpha - (\tfrac{1}{2}+a_k)T_h +$$

$$+ (c_k - e_k)(\tfrac{1}{2}+a_k)P_h\big] + (\beta_{1k}\xi_1 + \beta_{2k}\xi_2 + \beta_{3k}\xi_3)\big[T_\beta - (c_k - e_k)(P_\beta + T_z) +$$

$$+ P_z (c_k - e_k)^2\big]\Big]\Big\} \beta_{ik}\,\Delta x_k$$

or

(28)

$$Q_i = \omega^2 \sum_j C_{ij}\,\xi_j$$

where

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(29)

$$C_{ij} = \sum_k \pi\rho b_k^4 \Delta x_k \Big[\Big\{ \frac{h_{jk}}{b_k^2} L_h + \frac{\alpha_{jk}}{b_k}\left(L_\alpha - L_h(\tfrac{1}{2}+a_k)\right) + \frac{\beta_{jk}}{b_k}\left[L_\beta - (c_k - e_k)L_z\right]\Big\} h_{ik} + \frac{h_{jk}}{b_k}\left[M_h - (\tfrac{1}{2}+a_k)L_h\right] +$$

$$+ \alpha_{jk}\left[M_\alpha - (\tfrac{1}{2}+a_k)L_\alpha - M_h(\tfrac{1}{2}+a_k) + L_h(\tfrac{1}{2}+a_k)^2\right] +$$

$$+ \beta_{jk}\left[M_\beta - L_\beta(\tfrac{1}{2}+a_k) - M_z(c_k - e_k) + L_z(c_k - e_k)(\tfrac{1}{2}+a_k)\right]\Big\}\alpha_{ik} +$$

$$+ \Big\{ \frac{h_{jk}}{b_k}\left[T_h - (c_k - e_k)P_h\right] + \alpha_{jk}\Big[T_\alpha - P_\alpha(c_k - e_k) +$$

$$- T_h(\tfrac{1}{2}+a_k) + P_h(\tfrac{1}{2}+a_k)(c_k - e_k)\Big] + \beta_{jk}\Big[T_\beta - (P_\beta + T_z)(c_k - e_k)$$

$$+ P_z(c_k - e_k)^2\Big]\Big\}\beta_{ik}\Big]$$

Obvious simplifications occur when the quarter chord may be chosen as the reference axis $\left[(\tfrac{1}{2} + a_k) = 0\right]$ or there is no aerodynamic overhang on the aileron ($c_k - e_k = 0$). In a typical case both of these conditions may be assumed as well as the following simplified expressions for h_k, α_k, and β_k:

$$h_k = h_{1k}\xi_1 + h_{2k}\xi_2$$

$$\alpha_k = \alpha_{1k}\xi_1 + \alpha_{2k}\xi_2$$

$$\beta_k = \beta_{1k}\xi_1 + \beta_{2k}\xi_2$$

The general set of Lagrangian equations, with C_{ij} as defined above, may be written:

$$A_i \ddot{\xi}_i + \omega_i^2 A_i \xi_i = \omega^2 \sum_j C_{ij}\xi_j \qquad (i, j = 1, 2, 3)$$

where now the Q_i terms have replaced the zeros on the right side of (16). At the condition of flutter, harmonic motion at the frequency ω exists and thus the equations become, upon dividing by $-\omega^2$:

$$(30)\qquad A_i\left[1-\left(\frac{\omega_i}{\omega}\right)^2\right]\xi_i + \sum_j C_{ij}\xi_j = 0 \qquad (i,j = 1,2,3)$$

If damping g_i (as may be measured, for example, in ground vibration test) is introduced in the i^{th} mode, the equations take the form

$$\left\{A_1\left[1-\left(\frac{\omega_1}{\omega}\right)^2(1+ig_1)\right]+C_{11}\right\}\xi_1 + C_{12}\xi_2 + C_{13}\xi_3 = 0$$

$$C_{21}\xi_1 + \left\{A_2\left[1-\left(\frac{\omega_2}{\omega}\right)^2(1+ig_2)\right]+C_{22}\right\}\xi_2 + C_{23}\xi_3 = 0$$

$$C_{31}\xi_1 + C_{32}\xi_2 + \left\{A_3\left[1-\left(\frac{\omega_3}{\omega}\right)^2(1+ig_3)\right]+C_{33}\right\}\xi_3 = 0$$

The necessary and sufficient condition for the solution (other than the trivial $\xi_1 = \xi_2 = \xi_3 = 0$) of the above equations is that their determinant vanish. This stability determinant may be solved by a variety of methods well known in the field of flutter.

Case B - Aileron Motion Expressed as a Separate Degree of Freedom

Usually the aileron degree of freedom β is not measured during ground vibration test nor is it calculated during coupled mode analysis. If this is not done, then the aileron coordinate in the subsequent analysis cannot be expressed in terms of the generalized coordinates representing wing coupled motion. A separate coordinate must be provided for the aileron. Reference will be made in this case to results already developed under Case A.

For convenience in this case, only two generalized coordinates ξ_1 and ξ_2 will be assumed for the wing while the third ξ_3 will be reserved for the aileron and will no longer refer to the wing. Thus

$$h_k = h_{1k}\xi_1 + h_{2k}\xi_2$$

$$\alpha_k = \alpha_{1k}\xi_1 + \alpha_{2k}\xi_2$$

$$\beta_k = \beta_{3k}\xi_3$$

Also, a free body translation and pitch of the entire airplane can be introduced. These will be omitted for simplification, but the procedure will be similar to that illustrated here.

The kinetic energy T as expressed in equation (11) holds when h_{3k}, α_{3k}, β_{1k} and β_{2k} are set equal to zero. However, since only modes ξ_1 and ξ_2 are now to be considered orthogonal, the expression (12) for T is no longer valid but requires additional (cross-product) terms. The orthogonality condition (14) becomes

$$(14') \quad \sum_k \left\{ M_k h_{1k} h_{2k} + I_{\alpha k}\alpha_{1k}\alpha_{2k} + S_{\alpha k}(h_{1k}\alpha_{2k} + h_{2k}\alpha_{1k}) \right\} = 0$$

and the net result for T is, in this case:

$$(12') \quad T = \tfrac{1}{2}\left(A_1'\dot{\xi}_1^2 + A_2'\dot{\xi}_2^2 + A_3'\dot{\xi}_3^2\right) + A_{23}\dot{\xi}_2\dot{\xi}_3 + A_{13}\dot{\xi}_1\dot{\xi}_3$$

where

$$A_1' = \sum_k \left\{ M_k h_{1k}^2 + I_{\alpha k}\alpha_{1k}^2 + 2 S_{\alpha k} h_{1k}\alpha_{1k} \right\}$$

$$A_2' = \sum_k \left\{ M_k h_{2k}^2 + I_{\alpha k}\alpha_{2k}^2 + 2 S_{\alpha k} h_{2k}\alpha_{2k} \right\}$$

$$A_3' = \sum_k I_{\beta k}\beta_{2k}^2$$

$$A_{23} = \sum \left\{ S_{\beta k} h_{2k}\beta_{3k} + \left[I_{\beta k} + S_{\beta k}(c_k - a_k)b_k\right]\alpha_{2k}\beta_{3k} \right\}$$

$$A_{13} = \sum_k \left\{ S_{\beta k} h_{1k}\beta_{3k} + \left[I_{\beta k} + S_{\beta k}(c_k - a_k)b_k\right]\alpha_{1k}\beta_{3k} \right\}$$

(Note: β_{3k} is usually taken as unity for all k.)

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The potential energy as previously expressed in (15), is valid here, where $B_i = A_i' \omega_i^2$, giving:

(15') $$U = \frac{1}{2}\left(\omega_1^2 A_1' \xi_1^2 + \omega_2^2 A_2' \xi_2^2 + \omega_3^2 A_3' \xi_3^2\right)$$

The airforce expressions (21), (22), (23), are valid as they stand. The generalized forces Q_i in (26) become

$$Q_1 = \sum_k \left[L_k h_{1k} + \bar{M}_k \alpha_{1k} \right] \Delta x_k$$

$$Q_2 = \sum_k \left[L_k h_{2k} + \bar{M}_k \alpha_{2k} \right] \Delta x_k$$

$$Q_3 = \sum \bar{\bar{M}}_k \beta_{3k} \Delta x_k$$

and (27), (28), and (29) are valid with $h_{3k} = \alpha_{3k} = \beta_{1k} = \beta_{2k} = 0$

Selecting, for example, that case where the quarter chord is the reference axis and there is no aerodynamic overhang on the aileron ($\frac{1}{2} + a_k = 0$, $c_k - e_k = 0$ for all k) gives the following results for C_{ij}:

$$C_{11} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\frac{h_{1k}}{b_k} L_h + \alpha_{1k} L_\alpha \right] \frac{h_{1k}}{b_k} + \left[\frac{h_{1k}}{b_k} M_h + \alpha_{1k} M_\alpha \right] \alpha_{1k} \right\}$$

$$C_{12} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\frac{h_{2k}}{b_k} L_h + \alpha_{2k} L_\alpha \right] \frac{h_{1k}}{b_k} + \left[\frac{h_{2k}}{b_k} M_h + \alpha_{2k} M_\alpha \right] \alpha_{1k} \right\}$$

$$C_{13} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \beta_{3k} L_\beta \frac{h_{1k}}{b_k} + \beta_{3k} M_\beta \alpha_{1k} \right\}$$

$$C_{21} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\frac{h_{1k}}{b_k} L_h + \alpha_{1k} L_\alpha \right] \frac{h_{2k}}{b_k} + \left[\frac{h_{1k}}{b_k} M_h + \alpha_{1k} M_\alpha \right] \alpha_{2k} \right\}$$

$$C_{22} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\frac{h_{2k}}{b_k} L_h + \alpha_{2k} L_\alpha \right] \frac{h_{2k}}{b_k} + \left[\frac{h_{2k}}{b_k} M_h + \alpha_{2k} M_\alpha \right] \alpha_{2k} \right\}$$

$$C_{23} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\beta_{3k} L_\beta \frac{h_{2k}}{b_k} + \beta_{3k} M_\beta \alpha_{2k} \right] \right\}$$

$$C_{31} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\frac{h_{1k}}{b_k} T_h + \alpha_{1k} T_\alpha \right] \beta_{3k} \right\}$$

$$C_{32} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \left[\frac{h_{2k}}{b_k} T_h + \alpha_{2k} T_\alpha \right] \beta_{3k} \right\}$$

$$C_{33} = \sum_k \pi\rho b_k^4 \Delta x_k \left\{ \beta_{3k}^2 T_\beta \right\}$$

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The Lagrangian equations in this case lead to the equations below, using (12), (15), and (28) and writing: $\ddot{\xi}_i = -\omega^2 \xi_i$:

$$\left\{A_1'\left[1-\left(\frac{\omega_1}{\omega}\right)^2\right]+C_{11}\right\}\xi_1 + C_{12}\xi_2 + \left[A_{13}+C_{13}\right]\xi_3 = 0$$

$$C_{21}\xi_1 + \left\{A_2'\left[1-\left(\frac{\omega_2}{\omega}\right)^2\right]+C_{22}\right\}\xi_2 + \left[A_{23}+C_{23}\right]\xi_3 = 0$$

$$\left[A_{13}+C_{31}\right]\xi_1 + \left[A_{23}+C_{32}\right]\xi_2 + \left\{A_3'\left[1-\left(\frac{\omega_3}{\omega}\right)^2\right]+C_{33}\right\}\xi_3 = 0$$

A word of caution should be injected in reference to the methods of this appendix, for the case of control surfaces with large amounts of unbalance method A should be used, if the modes of vibration are obtained by ground vibration testing while method B can be used if the coupled modes are obtained by direct calculation. It has been found that the use of method B, when using ground vibration modes, in the case of highly unbalanced control surface may lead to appreciable errors in the flutter speed calculations.

REFERENCE

Srilg, B, and Wasserman, L., "Application of Three-Dimensional Flutter Theory to Aircraft Structures," AAF TR-4798, July, 1942.

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APPENDIX VI

TABLES OF AERODYNAMIC COEFFICIENTS FROM AAF TECHNICAL REPORT 4798

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AERODYNAMIC COEFFICIENTS FOR USE IN THREE DIMENSIONAL FLUTTER ANALYSES OF TAPERED AIRFOILS

$v/b\omega$	$K_2(L_h)$	$K_2(L_\alpha)$	$K_3(L_\alpha)$	$K_2(M_\alpha)$
0	0	0	0	0
.25	-.01525 - .25185j	-.01525 - .50185j	-.05296 + .00763j	-.25000j
.50	-.05770 - .51290j	-.05770 - 1.01290j	-.25645 + .02885j	-.50000j
.83	-.14617 - .88333j	-.14617 - 1.71666j	-.73610 + .12179j	-.83333j
1.25	-.29125 - 1.38525j	-.29125 - 2.63525j	-1.73156 + .36406j	-1.25000j
1.67	-.45933 - 1.92933j	-.45933 - 3.59600j	-3.21556 + .76555j	-1.66667j
2.00	-.60280 - 2.39160j	-.60280 - 4.39160j	-4.78320 + 1.20560j	-2.00000j
2.50	-.82500 - 3.12500j	-.82500 - 5.62500j	-7.81250 + 2.06250j	-2.50000j
2.94	-1.02235 - 3.80529j	-1.02235 - 6.74647j	-11.1920 + 3.00692j	-2.94118j
3.33	-1.19533 - 4.43333j	-1.19533 - 7.76666j	-14.7778 + 3.98444j	-3.33333j
3.75	-1.37983 - 5.10836j	-1.37983 - 8.85836j	-19.1539 + 5.17369j	-3.75000j
4.17	-1.55167 - 5.82416j	-1.55167 - 9.99083j	-24.2674 + 6.46527j	-4.16667j
5.00	-1.88600 - 7.27600j	-1.88600 - 12.2760j	-36.3800 + 9.43000j	-5.00000j
6.25	-2.34500 - 9.53500j	-2.34500 - 15.7850j	-59.5938 + 14.6562j	-6.25000j
8.33	-3.00167 - 13.4385j	-3.00167 - 21.7718j	-111.986 + 25.0138j	-8.33333j
10.00	-3.44600 - 16.6400j	-3.44600 - 26.6400j	-166.400 + 34.4600j	-10.00000j
12.50	-4.01000 - 21.5100j	-4.01000 - 34.0100j	-268.875 + 50.1250j	-12.50000j
16.67	-4.75333 - 29.7333j	-4.75333 - 46.4000j	-495.556 + 79.2222j	-16.66667j
	$K_1(L_h) = 1.0000$	$K_1(L_\alpha) = .5000$		$K_1(M_\alpha) = .3750$

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TABLE OF AERODYNAMIC COEFFICIENTS FOR WINGS AND AILERONS OF CONSTANT CHORD

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e = -.5

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.37922 0j	.54008 0j	.54008 0j	.37922 0j	.38883 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	.37060 -.23558j	.46533 -.44862j	.53693 -.05207j	.36305 -.26884j	.37001 -.25456j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	.34476 -.47117j	.24440 -.89373j	.52815 -.10604j	.31426 -.53679j	.31332 -.50838j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.59437j	.28351 -.78525j	-.30141 -1.44889j	.50986 -.18263j	.19770 -.88529j	.17743 -.83891j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	.16386 -1.17792j	-1.38614 -2.06339j	.47988 -.28640j	-.03900 -1.30290j	-.09362 -1.23563j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-.10367 -1.57059j	-2.95455 -2.57049j	.44512 -.39889j	-.35056 -1.69630j	-.48339 -1.61014j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.47117 0j	.80450 0j	.48994 0j	.70421 0j	.80450 0j	.47117 0j	.48994 0j
.25	.47117 -.03446j	.79013 -.23733j	.48697 -.06254j	.69859 -.13079j	.79854 -.09775j	.44059 -.39999j	.45123 -.37022j
.50	.47117 -.06892j	.75013 -.48332j	.47870 -.12687j	.68295 -.26497j	.78194 -.20054j	.34832 -.79826j	.33462 -.73907j
.83	.47117 -.11486j	.66673 -.83239j	.46145 -.21701j	.65035 -.45211j	.74734 -.34538j	.12620 -1.31276j	.05538 -1.21585j
1.25	.47117 -.17229j	.53009 -1.30537j	.43320 -.33725j	.59692 -.70038j	.69064 -.54163j	-.31975 -1.92178j	-.50068 -1.78082j
1.67	.47117 -.22972j	.37169 -1.81807j	.40044 -.46573j	.53498 -.96418j	.62491 -.75437j	-.56572 -2.18504j	-1.27963 -2.30321j

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e = -.5

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916 j	.37500 -2.00000j	-4.8560 -3.1860j	-.17211 -1.85467j	-4.57687 -2.89142j	.41546 -.49446j	-.73433 -1.99203j	-.88467 -1.89224j
2.50	.1752 -3.12500j	.37500 -2.50000j	-8.1375 -3.5625j	-.48224 -2.35584j	-7.65618 -3.22784j	.36952 -.64609j	-1.40658 -2.40321j	-1.64258 -2.28540j
2.94	-.0220 -3.80533j	.37500 -2.94118j	-11.7140 -3.7396j	-.81311 -2.77157j	-11.04000 -3.38078j	.32879 -.78674j	-2.14601 -2.73393j	-2.47155 -2.60256j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-1.15226 -3.14108j	-14.59416 -3.41017j	.29302 -.91659j	-2.92319 -3.00419j	-3.33896 -2.86246j
3.75	-.3793 -5.1084j	.37500 -3.75000j	-20.0357 -3.6847j	-1.55906 -3.53375j	-18.90480 -3.30736j	.25481 -1.05615j	-3.86611 -3.26180j	-4.38923 -3.11087j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-2.01372 -3.92642j	-23.89697 -3.14955j	.21921 -1.20414j	-4.95883 -3.50668	-5.59916 -3.34786j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.47117 -.27567j	.23646 -2.25368j	.37248 -.57373j	.48211 -1.18515j	.56881 -.93512j	-1.6348 -2.8997j	-2.1187 -2.6890j
2.50	.47117 -.34458j	.02708 -2.94479j	.32918 -.74357j	.40024 -1.53137j	.48193 -1.22188j	-2.9061 -3.4604j	-3.6647 -3.2102j
2.94	.47117 -.40539j	-.15856 -3.58585j	.29079 -.89988j	.32765 -1.84908j	.40490 -1.48787j	-4.3045 -3.8945j	-5.3526 -3.6138j
3.33	.47117 -.45944j	-.32199 -4.17767j	.25708 -1.04337j	.26391 -2.14007j	.33726 -1.73343j	-5.7743 -4.2355j	-7.1162 -3.9308j
3.75	.47117 -.51687j	-.49576 -4.81378j	.22106 -1.19734j	.19581 -2.45212j	.26499 -1.99737j	-7.5575 -4.5419j	-9.2502 -4.2153j
4.17	.47117 -.57430j	-.61750 -5.48830j	.18751 -1.35926j	.13237 -2.77918j	.19769 -2.27725j	-9.6241 -4.8243j	-11.7035 -4.4782j

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e = -.5

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.3750 -5.0000j	-37.7660 -2.8460j	-3.0666 -4.7117j	-35.6492 -2.4942j	.15015 -1.50431j	-7.5322 -3.9218j	-8.4406 -3.7518j
6.25	-1.3450 -9.5350j	.3750 -6.2500j	-61.4370 -1.1288j	-5.0049 -5.8896j	-57.9868 -.8594j	.05526 -1.971355j	-12.4262 -4.4001j	-13.8168 -4.22281j
8.33	-2.0020 -13.4385j	.3750 -8.3333j	-114.4920 3.2420j	-9.1925 -7.8528j	-108.0276 3.2775j	-.08058 -2.77840j	-23.3953 -4.8853j	-25.8000 -4.7159j
10.00	-2.4460 -16.6400j	.3750 -10.0000j	-169.3460 7.8200j	-13.4041 -9.4233j	-159.7489 7.6006j	-.17237 -3.44030j	-34.7363 -5.0499j	-38.1401 -4.9008j
12.50	-3.0100 -21.5100j	.3750 -12.5000j	-272.4100 16.1150j	-21.1572 -11.7792j	-256.9084 15.4262j	-.25898 -4.44717j	-56.0446 -5.0016j	-61.2593 -4.9009j
16.67	-3.7530 -29.7333j	.3750 -16.6666j	-499.8530 32.8222j	-37.9077 -15.7056j	-471.2866 31.1775j	-.44260 -6.14734	-103.0683 -4.3252j	-112.1315 -4.3411j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.47117 -.68917j	-.9727 -6.8564j	.12243 -1.68702j	.00931 -3.44077j	.0671 -2.8449j	-14.4908 -5.2477j	-17.4596 -4.8714j
6.25	.47117 -.86146j	-1.4053 -8.9852j	.03299 -2.19450j	-.15981 -4.46308j	-.1124 -3.7282j	-23.7462 -5.6101j	-28.3310 -5.2062j
8.33	.47117 -1.14861j	-2.0244 -12.6635j	-.09504 -3.06728j	-.40188 -6.21795j	-.3693 -5.2545j	-44.4907 -5.6240j	-52.5143 -5.2121j
10.00	.47117 -1.57833j	-2.4428 -1[illegible].6804j	-.18155 -3.78064j	-.56548 -7.65086j	-.5429 -6.5062j	-65.9386 -5.21[illegible]	-77.3812 -4.[illegible]205j
12.50	.47117 -1.72291j	-2.9743 -20.2696j	-.29146 -4.86438j	-.77328 -9.82518j	-.7634 -8.4104j	-106.2366 -4.0364j	-123.9195 -3.7087j
16.67	.47117 -2.29722j	-3.6744 -[illegible].0187j	-.43684 -6.69106j	-1.04704 -13.48834j	-1.0539 -11.6257j	-195.1469 -.9497j	-226.2114 -.7968j

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e = -.4

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.33285 0j	.46244 0j	.46244 0j	.33285 0j	.29928 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	.32191 -.22776j	.39035 -.41107j	.45984 -.04289j	.31953 -.23024j	.28329 -.20975j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	.28909 -.45552j	.17280 -.81803j	.45261 -.08734j	.27934 -.45972j	.23511 -.41881j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.59487j	.21130 -.75917j	-.35098 -1.32151j	.43754 -.15043j	.18260 -.75184j	.11952 -.69088j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	.05935 -1.13880j	-1.40263 -1.87026j	.41285 -.23590j	-.01162 -1.11713j	-.11112 -1.01725j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-.15340 -1.51843j	-2.92624 -2.31081j	.38423 -.32855j	-.29296 -1.45591j	-.44274 -1.32519j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.45552 0j	.74767 0j	.40683 0j	.63050 0j	.74767 0j	.45552 0j	.40683 0j
.25	.45552 -.04376j	.73360 -.23238j	.40443 -.05442j	.62573 -.12139j	.74249 -.08544j	.42899 -.37188j	.37173 -.32788j
.50	.45552 -.08752j	.69627 -.47326j	.39776 -.11028j	.61244 -.24565j	.72809 -.17399j	.34893 -.74225j	.26602 -.65437j
.83	.45552 -.14587j	.61277 -.81506j	.38385 -.18828j	.58474 -.41833j	.69807 -.29965j	.15621 -1.22176j	.01269 -1.07638j
1.25	.45552 -.21880j	.47898 -1.27818j	.36107 -.29189j	.53935 -.64638j	.64888 -.46992j	-.23067 -1.79155j	-.49192 -1.57661j
1.67	.45552 -.29174j	.32387 -1.78021j	.33466 -.40213j	.48673 -.88761j	.59186 -.65449j	-.79112 -2.32162j	-1.21555 -2.04013j

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e = -.4

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8860 -3.1860j	-.36732 -1.82208j	-4.50441 -2.57830j	.35979 -.40727j	-.58435 -1.71113j	-.78406 -1.55708j
2.50	.1752 -3.12500j	.37500 -2.50000j	-8.1375 -3.5625j	-.76116 -2.27761j	-7.50354 -2.83470j	.32195 -.53217j	-1.13805 -2.06741j	-1.42841 -1.88025j
2.94	-.0220 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-1.18136 -2.67954j	-10.80267 -2.91763j	.28840 -.64801j	-1.74711 -2.35532j	-2.13277 -2.14098j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-1.61206 -3.03678j	-14.27031 -2.88534j	.25894 -.75497j	-2.38724 -2.59174j	-2.86938 -2.35471j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-2.12868 -3.41641j	-18.47767 -2.71921j	.22747 -.86992j	-3.16391 -2.81858j	-3.76114 -2.55933j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.319[illegible] -3.5256j	-2.70608 -3.79604j	-23.35361 -2.49608j	.19815 -.99181j	-4.06394 -3.03493j	-4.78735 -2.75425j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.45552 -.35008j	.19146 -2.20675j	.31211 -.49456j	.44182 -1.08904j	.54318 -.81130j	-1.3716 -2.7145j	-1.9590 -2.3821j
2.50	.45552 -.43761j	-.01356 -2.88347j	.27719 -.63949j	.37227 -1.40371j	.46780 -1.06009j	-2.4746 -3.2507j	-3.3594 -2.8459j
2.94	.45552 -.51483j	-.19534 -3.51118j	.24624 -.77258j	.31060 -1.69175j	.40097 -1.29087j	-3.6878 -3.6711j	-4.8871 -3.2066j
3.33	.45552 -.58347j	-.35497 -4.09067j	.21905 -.89455j	.25645 -1.95507j	.34229 -1.50392j	-4.9630 -4.0059j	-6.4823 -3.4912j
3.75	.45552 -.65641j	-.52551 -4.71353j	.19001 -1.02536j	.19860 -2.23730j	.27959 -1.73291j	-6.5101 -4.3132j	-8.4120 -3.7489j
4.17	.45552 -.72934j	-.68437 -5.37400j	.16296 -1.16258j	.14471 -2.53228j	.22118 -1.97573j	-8.3031 -4.5996j	-10.6275 -3.9873j

e = .4

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.3750 -5.0000j	-37.7660 -2.8460j	-4.0432 -4.5552j	-34.8370 -1.7163j	.14127 -1.23905j	-6.1836 -3.4061j	-7.1963 -3.0873j
6.25	-1.3450 -9.5350j	.3750 -6.2500j	-61.4370 -1.1288j	-6.5047 -5.6940j	-56.6762 .0978j	.06310 -1.62374j	-10.2146 -3.8441j	-11.7506 -3.4771j
8.33	-2.0020 -13.4385j	.3750 -8.3333j	-114.4920 3.2420j	-11.8228 -7.5917j	-105.6275 4.5139j	-.04872 -2.28848j	-19.2495 -4.3170j	-21.8908 -3.8597j
10.00	-2.4460 -16.6400j	.3750 -10.0000j	-169.3460 7.8200j	-17.1714 -9.1104j	-156.2396 9.0450j	-.12439 -2.83368j	-28.5907 -4.5112j	-32.3241 -4.0498j
12.50	-3.0100 -21.5100j	.3750 -12.5000j	-272.4100 16.1150j	-27.0175 -11.3880j	-251.3350 17.1597j	-.22044 -3.66300j	-46.1418 -4.5594j	-51.8584 -4.0654j
16.67	-3.7530 -29.7333j	.3750 -16.6666j	-499.8530 32.8222j	-48.2899 -15.1841j	-461.1952 33.3439j	-.34696 -5.06338j	-84.8738 -4.1488j	-94.8126 -3.6385j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.45552 -.87521j	-.9926 -6.7136j	.11048 -1.44019j	.04016 -3.12857j	.10788 -2.46824j	-12.5255 -5.0498j	-[illegible]5.8233 -4.[illegible]908j
6.25	.45552 -1.03401j	-1.4161 -8.7980j	.03835 -1.86938j	-.10351 -4.04844j	-.04783 -3.23456j	-20.5554 -5.4864j	-25.6256 -4.6776j
8.33	.45552 -1.45868j	-2.0223 -12.3998j	-.06488 -2.60645j	-.30916 -5.62489j	-.27070 -4.55874j	-38.5532 -5.7075j	-47.4029 -4.7498j
10.00	.45552 -1.75042j	-2.4320 -15.3539j	-.13465 -3.20847j	-.44814 -6.91070j	-.42132 -5.64479j	-57.1613 -5.5159j	-69.7733 -4.4689j
12.50	.45552 -2.18503j	-2.9524 -19.8475j	-.22327 -4.12215j	-.62467 -8.86062j	-.61265 -7.29684j	-92.1237 -4.7442j	-111.6074 -3.6005j
16.67	.45552 -2.91737j	-3.63[illegible]0 -27.4352j	-.34002 -5.66171j	-.85724 -12.14387j	-.86469 -10.06644j	-169.2792 -2.4802j	-203.4848 -1.2509j

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e = -.3

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.28824 0j	.39064 0j	.39064 0j	.28824 0j	.22553 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.494233j	.27495 -.21607j	.32133 -.37338j	.38852 -.03493j	.27739 -.19452j	.21202 -.17015j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	.23510 -.43615j	.11192 -.74212j	.38264 -.07114j	.24466 -.38843j	.17127 -.33967j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.59487j	.14064 -.72688j	-.39320 -1.19401j	.37036 -.12252j	.16586 -.64111j	.07351 -.56017j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.04388 -1.09037j	-1.40966 -1.67751j	.35025 -.19214j	.00767 -.94039j	-.12160 -.8245j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-.30221 -1.45386j	-2.88519 -2.05249j	.32693 -.26761j	-.22149 -1.23238j	-.40205 -1.07378j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.43615 0j	.68808 0j	.33217 0j	.55736 0j	.68808 0j	.43615 0j	.33217 0j
.25	.43615 -.05324j	.67435 -.22683j	.33[illegible]26 -.04705j	.55333 -.11264j	.68361 -.07388j	.41321 -.54235j	.30031 -.28729j
.50	.43615 -.10628j	.63612 -.46194j	.32496 -.09525j	.5[illegible]212 -.2277[illegible]	.67116 -.15045j	.34398 -.68340j	.20431 -.57321j
.83	.43615 -.17713j	.55641 -.79556j	.31391 -.16230j	.51874 [illegible]	.64520 -.2[illegible]912j	.17735 -1.12[illegible]74j	-.02581 -.34276j
1.25	.43615 -.26569j	.42581 -1.24761j	.29579 -.25099j	.48043 -.59648j	.60266 -.40635j	-.1[illegible] -1.65309j	-.48407 -1.38172j
1.67	.43615 -.35426j	.27442 -1.73762j	.27479 -.3[illegible]j	.43602 -.81707j	.55375 -.56595j	-.64184 -2.14608j	-1.14100 -1.78715j

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$e = -.3$

$v/b\omega$	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8850 -3.1860j	-.56198 -1.74459j	-4.41567 -2.26769j	.30703 -.33172j	-.45882 -1.44964j	-.69054 -1.26144j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-1.04022 -2.18074j	-7.32768 -2.44635j	.27621 -.43345j	-.90982 -1.75380j	-1.23473 -1.52296j
2.94	-.0280 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-1.55047 -2.56558j	-10.53423 -2.46181j	.24888 -.52781j	-1.40990 -2.00066j	-1.82917 -1.73393j
3.33	-.1950 -4.43333j	.37500 -3.33333j	-15.4730 -3.7822j	-2.07347 -2.90768j	-13.90697 -2.37046j	.22489 -.61492j	-1.92728 -2.20415j	-2.45038 -1.90699j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0357 -3.6847j	-2.70080 -3.27111j	-18.00087 -2.14422j	.19925 -.70850j	-2.55987 -2.40059j	-3.20227 -2.07287j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-3.40284 -3.63457j	-22.74826 -1.89951j	.17537 -.80783j	-3.29296 -2.58845j	-4.06642 -2.23066j

$v/b\omega$	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.43615 -.42511j	.14518 -2.15396j	.25687 -.42347j	.39810 -1.00065j	.51126 -.70155j	-1.1438 -2.5135j	-1.7883 -2.0873j
2.50	.43615 -.53139j	-.05494 -2.81449j	.22911 -.54626j	.33940 -1.28661j	.44608 -.91669j	-2.0976 -3.0187j	-3.0845 -2.4952j
2.94	.43615 -.62516j	-.23237 -3.42718j	.20450 -.65875j	.28735 -1.54770j	.38829 -1.11624j	-3.1467 -3.4190j	-4.4677 -2.8134j
3.33	.43615 -.70851j	-.38818 -3.99282j	.18289 -.76166j	.24165 -1.78593j	.33754 -1.30047j	-4.2494 -3.7411j	-5.9108 -3.0654j
3.75	.43615 -.79708j	-.55464 -4.60077j	.15980 -.87196j	.19282 -2.04111j	.28332 -1.49849j	-5.5872 -4.0414j	-7.6560 -3.2954j
4.17	.43615 -.88564j	-.70971 -5.24545j	.13829 -.98737j	.14733 -2.30705j	.23282 -1.70846j	-7.1376 -4.3237j	-9.6569 -3.5081j

2/5/6

e = -3

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-5.08561 -4.36148j	-33.9338 -.96314j	.12804 -1.00921j	-5.01941 -2.91409j	-6.09480 -2.50087j
6.25	-1.3450 -9.5350j	.37500 -6.2500j	-61.4370 -1.1288j	-8.01465 -5.45185j	-55.8189 1.0160j	.06538 -1.38254j	-8.30267 -3.30575j	-9.92335 -2.61818j
8.33	-2.0080 -13.4385j	.37500 -8.33333j	-114.4980 3.8420j	-14.47844 -7.86910j	-102.9536 5.6836j	-.02575 -1.86397j	-15.30267 -3.74923j	-12.43864 -3.15714j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-20.96716 -8.72296j	-152.3843 10.3974j	-.08734 -2.30804j	-23.30336 -3.95401j	-27.19178 -3.29405j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-32.92332 -10.90370j	-245.1058 18.7592j	-.16557 -2.98352j	-37.56547 -4.06313j	-43.56832 -3.31682j
16.67	-3.7530 -29.7333j	.37500 -16.66666j	-499.8530 32.8882j	-58.75454 -14.53830j	-449.8943 35.2992j	-.26862 -4.12413j	-69.11272 -3.84522j	-79.55070 -2.99481j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.43615 -1.06277j	-1.0105 -6.5530j	.09657 -1.22069j	.05909 -2.84430j	.13484 -2.13434j	-10.7680 -4.7822j	-14.3467 -3.8377j
6.25	.43615 -1.32846j	-1.4239 -8.5876j	.03923 -1.58083 j	-.06217 -3.67161j	.00020 -2.79800j	-17.7325 -5.2654j	-23.1837 -4.1499j
8.33	.43615 -1.77128j	-2.0156 -12.1032j	-.04285 -2.19836j	-.23575 -5.08706j	-.19253 -3.94205j	-33.2956 -5.6280j	-42.7888 -4.2580j
10.00	.43615 -2.12554j	-2.4195 -14.9866j	-.09831 -2.70222j	-.35305 -6.24022j	-.32277 -4.88118j	-49.3865 -5.6009j	-62.9058 -4.0603j
12.50	.43615 -2.65693j	-2.9235 -19.3727j	-.16877 -3.46646j	-.50205 -7.98785j	-.48821 -6.30974j	-79.6193 -5.2413j	-100.4952 -3.3651j
16.67	.43615 -3.54257j	-3.5926 -26.7789j	-.26159 -4.75354j	-.69835 -10.92876j	-.70616 -8.72198j	-146.3374 -3.5299j	-182.9782 -1.4959j

21516

e = -2

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.24575 0j	.32490 0j	.32490 0j	.24575 0j	.16585 0j
.25	.9648 -.2519j	.37500 -.25000j	.42179 -.49423j	.23015 -.20652j	.25849 -.33579j	.32320 -.02807j	.23703 -.16180j	.15451 -.13554j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	.18357 -.41304j	.03763 -.66646j	.31847 -.05717j	.21073 -.32312j	.12033 -.27052j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.59487j	.072497 -.68837j	-.42779 -1.06720j	.30860 -.09846j	.14741 -.53349j	.03825 -.44601j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.14410 -1.03259j	-1.40683 -1.48640j	.29244 -.15441j	.02028 -.78674j	-.12550 -.65626j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-.44734 -1.37682j	-2.83081 -1.79726j	.27370 -.21505j	-.16387 -1.02694j	-.3[illegible]71 -.85443j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.41304 0j	.62547 0j	.26606 0j	.45584 0j	.62647 0j	.41304 0j	.26606 0j
.25	.41304 -.06238j	.61321 -.22061j	.26457 -.04032j	.46247 -.10428j	.62262 -.06352j	.39332 -.31178j	.25716 -.24861j
.50	.41304 -.12475j	.57593 -.44929j	.26043 -.08154j	.47310 -.21058j	.61192 -.12936j	.33380 -.62245j	.15002 -.49592j
.83	.41304 -.20792j	.49840 -.77377j	.25179 -.13869j	.45354 -.35726j	.58960 -.22279j	.19052 -1.0260j	-.05[illegible] -.81553j
1.25	.41304 -.31188j	.37139 -1.21344j	.23763 -.21392j	.42151 -.54921j	.55303 -.34937j	[illegible] -1.50046j	-.47466 -1.19448j
1.67	.41304 -.41584j	.22414 -1.69004j	.22122 -.29326j	.38437 -.75047j	.51063 -.48660j	-.51381 -1.96140j	-1.07022 -1.54610j

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e = -2

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v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8860 -3.1860j	-.75227 -1.65215j	-4.30989 -1.96171j	.25771 -.26658j	-.35459 -1.20886j	-.60248 -1.00360j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-1.31365 -2.06519j	-7.12745 -2.06552j	.23294 -.34853j	-.71702 -1.46427j	-1.05813 -1.21139j
2.94	-.0220 -3.8053j	.37500 -2.94118j	-11.7240 -3.7396j	-1.91259 -2.42963j	-10.23317 -2.01663j	.21098 -.42415j	-1.11568 -1.67232j	-1.55536 -1.37910j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-2.52652 -2.75355j	-13.50[illegible]7 -1.86944j	.19170 -.49416j	-1.53466 -1.84447j	-2.07456 -1.51660j
3.75	-.3798 -5.1064j	.37500 -3.75000j	-20.0337 -3.6647j	-3.26290 -3.09777j	-17.47165 -1.58687j	.17110 -.56940j	-2.04302 -2.01146j	-2.70278 -1.64862j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-4.08592 -3.44198j	-22.07763 -1.24473j	.15191 -.64918j	-2.63214 -2.17158j	-3.42376 -1.77401j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.41304 -.49901j	.09843 -2.09498j	.20720 -.35937j	.35266 -.91744j	.47444 -.60319j	-.94536 -2.30059j	-1.6610 -1.8061j
2.50	.41304 -.62376j	-.09621 -2.73742j	.18551 -.46845j	.30357 -1.17674j	.41840 -.78816j	-1.76542 -2.76981j	-2.8293 -2.1601j
2.94	.41304 -.73384j	-.26878 -3.33334j	.16627 -.55663j	.26005 -1.41287j	.36871 -.95973j	-2.66745 -3.14468j	-4.0796 -2.4371j
3.33	.41304 -.83168j	-.42032 -3.88348j	.14938 -.64263j	.22185 -1.62790j	.32508 -1.11813j	-3.71551 -3.44894j	-5.3828 -2.6571j
3.75	.41304 -.93564j	-.58223 -4.47479j	.13134 -.73476j	.18100 -1.85809j	.27846 -1.28838j	-4.76578 -3.73526j	-6.9581 -2.8990j
4.17	.41304 -1.03960j	-.73149 -5.10181j	.11453 -.83087j	.14296 -2.09730j	.23504 -1.46891j	-6.09877 -4.00800j	[illegible].7615 -3.04599j

21516

e = -.2

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-5.99185 -4.13037j	-32.93477 -.24113j	.11468 -.81101j	-4.01953 -2.45155j	-5.11465 -1.98920j
6.25	-1.3450 -9.5750j	.37500 -6.2500j	-61.4370 -1.1288j	-9.50050 -5.16296j	-53.60660 1.88661j	.06351 -1.06281j	-6.65799 -2.79373j	-8.30375 -2.24258j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 2.2420j	-17.08090 -6.88392j	-99.99071 6.77415j	-.00972 -1.49791j	-12.57172 -3.19585j	-15.38579 -2.51546j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-24.70465 -8.26074j	-147.98009 11.64214j	-.05921 -1.85476j	-18.68597 -3.39701j	-22.65775 -2.62697j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-38.73925 -10.32993j	-238.18363 20.20436j	-.12207 -2.39759j	-30.17389 -3.53959j	-36.25198 -2.65375j
16.67	-3.7530 -29.7333j	.37500 -16.66666j	-499.8530 32.8222j	-69.06091 -13.76793j	-473.31521 37.01416j	-.20489 -3.31420j	-55.52560 -3.45995j	-66.09574 -2.41539j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.41304 -1.24752j	-1.0256 -6.3730j	.08191 -1.02507j	.06917 -2.53017j	.15080 -1.63908j	-9.2580 -4.4604j	-12.9858 -3.3392j
6.25	.41304 -1.55940j	-1.4277 -8.3524j	.03710 -1.38430j	-.03825 -3.32442j	.03504 -2.40482j	-15.2081 -4.9629j	-20.9351 -3.6221j
8.33	.41304 -2.07920j	-2.0032 -11.7718j	-.02705 -1.83653j	-.17739 -4.59009j	-.13066 -3.38932j	-28.5891 -5.4200j	-38.5440 -3.7554j
10.00	.41304 -2.49904j	-2.3981 -14.5762j	-.07040 -2.25400j	-.27548 -5.62163j	-.24265 -4.19677j	-42.4238 -5.5129j	-56.5916 -3.6205j
12.50	.41304 -3.11880j	-2.8862 -18.8422j	-.12547 -2.80683j	-.40009 -7.18389j	-.38489 -5.42504j	-68.4176 -5.2921j	-90.2845 -3.1002j
16.67	.41304 -4.15840j	-3.5370 -26.0456j	.19797 -3.98196j	-.56425 -9.81124j	-.57228 -7.49904j	-125.7810 -4.1972j	-164.1490 -1.5933j

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e = -1

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.20575 0j	.26539 0j	.26539 0j	.20575 0j	.11571 0j
.25	.9848 -.25193j	.37500 -.25000j	.42179 -.49423j	.18794 -.19315j	.20202 -.29855j	.26404 -.02219j	.19886 -.13218j	.10629 -.10571j
.50	.9423 -.51293j	.37500 -.50000j	.18580 -.98405j	.13449 -.38629j	.01012 -.59155j	.26030 -.04520j	.17806 -.26398j	.07788 -.21093j
.83	.8538 -.68333j	.37500 -.33333j	-.35230 -1.59487j	.00782 -.64380j	-.45455 -.94191j	.25250 -.07784j	.12800 -.43598j	.00967 -.34767j
1.25	.7088 -1.38533j	.37500 -1.25000j	-1.52280 -2.27119j	-.23963 -.96574j	-1.39381 -1.29824j	.23973 -.12208j	.02749 -.64331j	-.12634 -.51140j
1.67	.5407 -1.92933j	.37500 -1.66667j	-3.17490 -2.83045j	-.58607 -1.28767j	-2.76268 -1.54695j	.22491 -.17002j	-.11811 -.54031j	-.32152 -.66565j

v/bω	M_z	L_z	T_z	P_z	R_z	P_α	P_β
0	.38630 0j	.56356 0j	.20846 0j	.41690 0j	.56356 0j	.38630 0j	.20846 0j
.25	.38630 -.07126j	.55042 -.21372j	.20732 -.03427j	.44411 -.09614j	.56027 -.05439j	.36947 -.28054j	.18229 -.21206j
.50	.38630 -.14254j	.51459 -.43525j	.20415 -.06902j	.40636 -.19396j	.55114 -.11036j	.31869 -.56012j	.10558 -.42292j
.83	.38630 -.23754j	.43949 -.74999j	.19753 -.11717j	.39020 -.32846j	.53210 -.19007j	.19445 -.92381j	-.08575 -.69538j
1.25	.38630 -.35630j	.31645 -1.17552j	.18668 -.18026j	.36373 -.50371j	.50099 -.29806j	-.04895 -1.35960j	-.46201 -1.01844j
1.67	.38630 -.47507j	.17380 -1.65723j	.17411 -.24652j	.33303 -.68665j	.46473 -.41513j	-.40443 -1.77030j	-1.00035 -1.31833j

21516

$e = -.1$

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8860 -3.1860j	-.93442 -1.54518j	-4.18645 -1.66260j	.21227 -.21076j	-.26509 -.98982j	-.52195 -.78171j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-1.57577 -1.93147j	-6.90192 -1.69516j	.19268 -.27539j	-.55544 -1.20025j	-.89925 -.94338j
2.94	-.0220 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-2.26003 -2.27232j	-9.89814 -1.58570j	.17532 -.33535j	-.87063 -1.37227j	-1.31052 -1.07384j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-2.96139 -2.57327j	-13.05395 -1.38643j	.16008 -.39069j	-1.20189 -1.51505j	-1.73953 -1.18083j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6647j	-3.80267 -2.89721j	-16.88778 -1.05197j	.14379 -.45018j	-1.60381 -1.65417j	-2.25842 -1.28368j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-4.74292 -3.21912j	-21.33895 -.65723j	.12862 -.51326j	-2.06958 -1.78967j	-2.85299 -1.38122j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.38630 -.57009j	.05203 -2.02951j	.16338 -.30152j	.30683 -.83792j	.45385 -.51460j	-.77261 -2.07907j	-1.5519 -1.5403j
2.50	.38630 -.71261j	-.13654 -2.65187j	.14676 -.38703j	.26626 -1.07214j	.38604 -.67241j	-1.47224 -2.50847j	-2.5876 -1.8429j
2.94	.38630 -.83836j	-.30372 -3.22916j	.13003 -.46497j	.23029 -1.28487j	.34365 -.81873j	-2.24170 -2.85396j	-3.7139 -2.0803j
3.33	.38630 -.95014j	-.45052 -3.76211j	.11909 -.53598j	.19870 -1.47821j	.30643 -.95392j	-3.05061 -3.13639j	-4.8865 -2.2692j
3.75	.38630 -1.06891j	-.60737 -4.33694j	.10527 -.61202j	.16495 -1.68506j	.26666 -1.09917j	-4.03194 -3.40573j	-6.3036 -2.4436j
4.17	.38630 -1.18768j	-.75347 -4.94237j	.09240 -.69111j	.13352 -1.89935j	.22961 -1.25318j	-5.16917 -3.66179j	-7.9231 -2.6051j

$e = -.1$

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.3750 -5.0000j	-37.7660 -2.8460j	-6.9203 -3.8630j	-31.8357 .4429j	.09918 -.64121j	-3.1665 -2.0234j	-4.2466 -1.5489j
6.25	-1.3450 -9.5350j	.3750 -6.2500j	-61.4370 -1.1238j	-10.9288 -4.8287j	-51.8525 2.7012j	.05875 -.84088j	-5.2525 -2.3152j	-6.8713 -1.7469j
8.33	-2.0020 -13.4385j	.3750 -8.3333j	-114.4920 3.2420j	-19.5889 -6.4382j	-96.7255 7.7726j	.00083 -1.18425j	-9.9280 -2.6686j	-12.6909 -1.9616j
10.00	-2.4460 -16.6400j	.3750 -10.0000j	-169.3460 7.8200j	-28.2966 -7.7259j	-145.1870 12.7632j	-.03830 -1.46642j	-14.7621 -2.8561j	-18.6595 -2.0512j
12.50	-3.0100 -21.5100j	.3750 -12.5000j	-272.4100 16.1150j	-44.3323 -9.6574j	-230.5359 21.4739j	-.06800 -1.89559j	-23.8448 -3.0114j	-29.8071 -2.0775j
16.67	-3.7530 -29.7333j	.3750 -16.6666j	-499.8530 32.8222j	-78.9729 -12.8765j	-423.3972 38.4586j	-.15348 -2.62028j	-43.8884 -3.0162j	-54.2569 -1.9041j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.38630 -1.42522j	-1.0569 -6.2744j	.06742 -.85079j	.07253 -2.33163j	.15774 -1.56558j	-7.8474 -4.0962j	-11.7141 -2.8631j
6.25	.38630 -1.78152j	-1.4264 -8.0914j	.03309 -1.09639j	-.01128 -2.99487j	.05898 -2.05165j	-12.9407 -4.5977j	-18.8320 -3.1142j
8.33	.38630 -2.37536j	-1.9839 -11.4039j	-.01604 -1.51609j	-.13124 -4.12557j	-.08239 -2.89156j	-24.3565 -5.1088j	-34.5927 -3.2548j
10.00	.38630 -2.85043j	-2.3607 -14.1207j	-.04925 -1.85773j	-.21232 -5.04451j	-.17792 -3.58043j	-36.1595 -5.2852j	-50.7198 -3.1665j
12.50	.38630 -3.56304j	-2.8393 -18.2533j	-.09142 -2.37526j	-.31530 -6.43528j	-.29928 -4.66883j	-58.3357 -5.2422j	-80.7993 -2.7704j
16.67	.38630 -4.75071j	-3.4698 -25.2316j	-.14693 -3.24579j	-.45096 -8.77271j	-.45915 -6.39772j	-107.2743 -4.5504j	-146.6778 -1.5795j

2/5/6

e = 0 VI-18

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.16861 0j	.21221 0j	.212210 0j	.16861 0j	.08191 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	.14871 -.17805j	.15206 -.26189j	.211168 -.017204j	.16327 -.10571j	.07419 -.08039j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	.08903 -.35610j	-.03048 -.51788j	.208269 -.035036j	.14715 -.21113j	.05087 -.16038j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.59487j	-.05243 -.59348j	-.47324 -.81900j	.202223 -.060340j	.10834 -.34878j	-.00907 -.26427j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.32876 -.89025j	-1.37027 -1.11434j	.192318 -.094626j	.03043 -.51490j	-.11655 -.38862j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-.71563 -1.18702j	-2.67968 -1.30335j	.180835 -.131792j	-.08243 -.67302j	-.27632 -.50571j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.35610 0j	.50000 0j	.15916 0j	.35132 0j	.50000 0j	.35610 0j	.15916 0j
.25	.35610 -.07958j	.48752 -.20609j	.15830 -.02854j	.34905 -.08811j	.49818 -.04576j	.34189 -.24895j	.13555 -.17782j
.50	.35610 -.15916j	.45278 -.41971j	.15593 -.05759j	.34274 -.17758j	.48952 -.09319j	.29901 -.49711j	.06440 -.35457j
.83	.35610 -.26526j	.38036 -.72284j	.15098 -.09757j	.32958 -.30019j	.47344 -.16049j	.19580 -.82028j	-.10610 -.58291j
1.25	.35610 -.39789j	.26171 -1.13356j	.14288 -.14972j	.30803 -.45926j	.44709 -.25169j	-.01142 -1.20843j	-.44506 -.85365j
1.67	.35610 -.53052j	.12415 -1.57879j	.13348 -.20423j	.28303 -.62499j	.41655 -.35054j	-.31159 -1.57530j	-.92928 -1.10507j

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e = 0

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.39163j	.37500 -2.00000j	-4.8860 -3.1860j	-1.10464 -1.42440j	-4.04451 -1.37262j	.17103 -.16337j	-.19931 -.79324j	-.44019 -.59376j
2.50	.1752 -3.12500j	.37500 -2.50000j	-8.1375 -3.5625j	-1.82084 -1.78050j	-6.64970 -1.33821j	.15585 -.21347j	-.42142 -.96886j	-.74827 -.71642j
2.94	-.0220 -3.80533j	.37500 -2.94118j	-11.7140 -3.7396j	-2.58495 -2.09471j	-9.52713 -1.17267j	.14240 -.25994j	-.66573 -1.10192j	-1.08365 -.81540j
3.33	-.1990 -4.43333j	.37500 -3.33333j	-15.4730 -3.7822j	-3.36818 -2.37398j	-12.55991 -.92567j	.13058 -.30284j	-.92251 -1.21769j	-1.43310 -.89655j
3.75	-.3793 -5.10844j	.37500 -3.75000j	-20.0337 -3.6847j	-4.30763 -2.67075j	-16.24579 -.54445j	.11795 -.34895j	-1.23405 -1.33095j	-1.85557 -.97469j
4.17	-.5580 -5.82423j	.37500 -4.16667j	-25.3790 -3.5256j	-5.35762 -2.96750j	-20.52774 -.10869j	.10619 -.39785j	-1.59509 -1.44000j	-2.33883 -1.04868j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.35610 -.63662j	.00672 -1.95707j	.12546 -.24935j	.26170 -.76086j	.39048 -.43453j	-.62248 -1.85210j	-1.4248 -1.2913j
2.50	.35610 -.79578j	-.17511 -2.55722j	.11304 -.31926j	.22866 -.97123j	.35011 -.56778j	-1.21324 -2.23881j	-2.3541 -1.5455j
2.94	.35610 -.93621j	-.33631 -3.11391j	.10203 -.38280j	.19937 -1.14278j	.31431 -.69138j	-1.86306 -2.55185j	-3.3634 -1.7453j
3.33	.35610 -1.06103j	-.47788 -3.62784j	.09236 -.44058j	.17365 -1.33461j	.28288 -.80549j	-2.54602 -2.80926j	-4.4129 -1.9047j
3.75	.35610 -1.19366j	-.62913 -4.18022j	.08702 -.50241j	.14617 -1.51942j	.24930 -.92814j	-3.37467 -3.05679j	-5.6808 -2.0585j
4.17	.35610 -1.32629j	-.77002 -4.76997j	.07240 -.56653j	.12057 -1.71027j	.21802 -1.05829j	-4.33495 -3.29314j	-7.1272 -2.2893j

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e = 0

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-7.78915 -3.56100j	-30.6297 1.0816j	.083377 -.497023j	-2.44534 -1.63341j	-3.47067 -1.17611j
6.25	-1.3450 -9.5350j	.37500 -6.25000j	-61.4370 -1.1288j	-12.26538 -4.45125j	-49.8852 3.4495j	.052023 -.651336j	-4.06231 -1.87586j	-5.59924 -1.32686j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-21.93624 -5.93498j	-93.1364 8.6652j	.007144 -.917984j	-7.68650 -2.17687j	-10.31070 -1.49140j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-31.66240 -7.12200j	-137.9130 13.7423j	-.023186 -1.136678j	-11.44997 -2.34382j	-15.13646 -1.56125j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-49.56734 -8.90250j	-222.1103 22.5434j	-.061713 -1.469348j	-18.47385 -2.49669j	-24.14094 -1.58504j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-88.25084 -11.87000j	-408.0434 39.5972j	-.112467 -2.031084j	-34.01051 -2.55460j	-43.87011 -1.46177j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.35610 -1.59155j	-1.0433 -5.9540j	.05373 -.69587j	.07091 -2.09500j	.15733 -1.32198j	-6.5965 -3.7002j	-10.5106 -2.4082j
6.25	.35610 -1.98944j	-1.4189 -7.8026j	.02807 -.89443j	.00267 -2.68416j	.07394 -1.73241j	-10.8972 -4.1840j	-16.8588 -2.6292j
8.33	.35610 -2.65258j	-1.9566 -10.9969j	-.00865 -1.23311j	-.09501 -3.68670j	-.04543 -2.44164j	-20.5368 -4.7161j	-30.8736 -2.7669j
10.00	.35610 -3.18310j	-2.3199 -13.6167j	-.03347 -1.50846j	-.16103 -4.50044j	-.12610 -3.02332j	-30.5032 -4.9454j	-45.2011 -2.7127j
12.50	.35610 -3.97888j	-2.7814 -17.6019j	-.06500 -1.92525j	-.24488 -5.73112j	-.22858 -3.90815j	-49.2289 -5.0296j	-71.8975 -2.4159j
16.67	.35610 -5.30517j	-3.3894 -24.3311j	-.10653 -2.62590j	-.35535 -7.79809j	-.36357 -5.40225j	-90.5530 -4.6469j	-130.3066 -1.4881j

e = .1

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.13463 0j	.16539 0j	.16539 0j	.13463 0j	.05434 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	.11289 -.16137j	.10853 -.22607j	.16460 -.01302j	.13059 -.08240j	.04810 -.05930j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	.04766 -.32274j	-.06409 -.44594j	.16241 -.02651j	.11839 -.16457j	.02929 -.11329j
.83	.8536 -.8833j	.37500 -.83333j	-.38230 -1.59487j	-.10694 -.53788j	-.48369 -.69929j	.15783 -.04565j	.08900 -.27193j	-.01584 -.19488j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.40895 -.80685j	-1.33576 -.93600j	.15034 -.07159j	.03009 -.40164j	-.10566 -.28648j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-.83177 -1.07582j	-2.58251 -1.06834j	.14165 -.09971j	-.05529 -.52529j	-.23420 -.37269j

v/bω	M_s	L_s	T_s	P_s	P_h	P_α	P_β
0	.32274 0j	.43644 0j	.11983 0j	.28979 0j	.43644 0j	.32274 0j	.11983 0j
.25	.32274 -.08697j	.42447 -.19766j	.11922 -.02534j	.28797 -.08008j	.43413 -.03813j	.31090 -.21735j	.09869 -.14608j
.50	.32274 -.17395j	.39116 -.40254j	.11749 -.04718j	.28293 -.16125j	.42770 -.07765j	.27517 -.43404j	.03494 -.29123j
.83	.32274 -.28991j	.32170 -.69326j	.11390 -.07979j	.27241 -.27213j	.41431 -.13374j	.18916 -.71654j	-.11775 -.47871j
1.25	.32274 -.43486j	.20790 -1.08719j	.10802 -.12212j	.25518 -.41537j	.39235 -.20973j	.01649 -1.05647j	-.42090 -.70098j
1.67	.32274 -.57982j	.07597 -1.51420j	.10120 -.16617j	.23521 -.56362j	.36690 -.29210j	-.23364 -1.37869j	-.85349 -.90746j

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e = .1

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.0000j	-4.8860 -3.1860j	-1.25693 -1.29096j	-3.88285 -1.09410j	.13484 -.12360j	-.14372 -.61948j	-.36588 -.43751j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1575 -3.5625j	-2.03968 -1.61370j	-6.36880 -.99780j	.12275 -.16150j	-.31176 -.75264j	-.61304 -.52778j
2.94	-.0220 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-2.87481 -1.89847j	-9.11730 -.78121j	.11257 -.19666j	-.49596 -.86211j	-.88171 -.60063j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7882j	-3.73081 -2.15158j	-12.01615 -.49172j	.10363 -.22911j	-.69086 -.95345j	-1.16131 -.66036j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-4.75757 -2.42055j	-15.54071 -.06972j	.09408 -.26400j	-.92655 -1.04521j	-1.49915 -.71793j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-5.90514 -2.68952j	-19.63766 .41269j	.08518 -.30099j	-1.19970 -1.12974j	-1.88488 -.77236j

v/bω	M_z	L_z	T_z	R_z	R_h	P_α	P_β
2.00	.32274 -.69578j	-.03666 -1.87770j	.09538 -.20250j	.21816 -.68541j	.34518 -.36209j	-.49270 -1.62254j	-1.2955 -1.0505j
2.50	.32274 -.86973j	-.21104 -2.45260j	.08637 -.25863j	.19176 -.87286j	.32154 -.47313j	-.98498 -1.98460j	-2.1233 -1.2696j
2.94	.32274 -1.02321j	-.36566 -2.98651j	.07838 -.30949j	.16835 -1.04221j	.28171 -.57622j	-1.52846 -2.84290j	-3.0210 -1.4342j
3.33	.32274 -1.15963j	-.50143 -3.47941j	.07136 -.35565j	.14780 -1.19549j	.29552 -.67121j	-2.09558 -2.47293j	-3.9535 -1.5658j
3.75	.32274 -1.30499j	-.64649 -4.00919j	.06387 -.40501j	.12583 -1.35930j	.22753 -.77341j	-2.78607 -2.69570j	-5.0792 -1.6884j
4.17	.32274 -1.44954j	-.75162 -4.57098j	.05688 -.45602j	.10558 -1.52794j	.20147 -.88178j	-3.58625 -2.90915j	-6.3613 -1.8018j

e = .1 VI-23

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-8.56262 -3.22740j	-29.3073 1.6671j	.06792 -.37602j	-1.84296 -1.28413j	-2.78756 -.86627j
6.25	-1.3450 -9.5350j	.37500 -6.25000j	-61.4370 -1.1288j	-13.45482 -4.03425j	-47.7489 4.1206j	.04420 -.49277j	-3.06627 -1.47965j	-4.48242 -.97757j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4930 3.8420j	-24.02438 -5.37898j	-89.1931 9.4349j	.01025 -.69450j	-5.80815 -1.72754j	-8.22687 -1.09977j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-34.65437 [illegible]	-132.1124 14.5584j	-.01270 [illegible]	-8.64301 -1.86997j	-12.05677 -1.15247j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-54.22318 -8.06850j	-212.8328 [illegible]	[illegible] [illegible]	-13.96936 -2.00981j	-19.19559 -1.17255j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-96.50148 -10.75808j	-391.1164 40.3882j	-.08025 -1.53662j	-25.72362 -2.09392j	-34.81994 -1.08749j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.32274 -1.73945j	-1.0437 -5.7104j	.04334 -.55886j	.06569 -1.36764j	.15090 -1.10159j	-5.4707 -3.2813j	-9.3577 -1.9847j
6.25	.32274 -2.17431j	-1.4040 -7.4834j	.02269 -.71643j	.01115 -2.38683j	.08141 -1.44360j	-9.0545 -3.7340j	-14.9709 -2.1725j
8.33	.32274 -2.89908j	-1.9196 -10.5469j	-.00193 -.98465j	-.06693 -3.26859j	-.01806 -2.03459j	-17.0871 -4.2598j	-27.3406 -2.3001j
10.00	.32274 -3.47890j	-2.2631 -13.0595j	-.01994 -1.20242j	-.11968 -3.98337j	-.08528 -2.51930j	-25.3920 -4.5170j	-39.9686 -2.2701j
12.50	.32274 -4.34863j	-2.7107 -16.88173j	-.04281 -1.53182j	-.18670 -5.06358j	-.17067 -3.25663j	-40.9958 -4.6663j	-63.4743 -2.0524j
16.67	.32274 -5.79817j	-3.2939 -23.3356j	-.07295 -2.08515j	-.27499 -6.87661j	-.28316 -4.5016j	-75.4307 -4.5323j	-114.8470 -1.3422j

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$\theta = .2$

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.10415 0j	.12490 0j	.12490 0j	.10415 0j	.03429 0j
.25	.9848 -.25193j	.37500 -.25000j	.42179 -.49423j	.08076 -.14328j	.07154 -.19131j	.12432 -.00955j	.10118 -.06223j	.02936 -.04214j
.50	.9423 -.51293j	.37500 -.50000j	.18580 -.98405j	.01059 -.28657j	-.09065 -.37624j	.12271 -.01945j	.09223 -.12430j	.01451 -.08403j
.83	.8538 -.88333j	.37500 -.83333j	-.38230 -1.59487j	-.15573 -.47759j	-.46571 -.58364j	.11936 -.03351j	.07069 -.20545j	-.02111 -.13841j
1.25	.7088 -1.38533j	.37500 -1.25000j	-1.52280 -2.27119j	-.48063 -.71641j	-1.28979 -.76464j	.11386 -.05254j	.02743 -.30357j	-.09079 -.20741j
1.67	.5407 -1.92933j	.37500 -1.66667j	-3.17490 -2.83045j	-.93550 -.95524j	-2.46867 -.84393j	.10742 -.07318j	-.03524 -.39726j	-.19707 -.26455j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.28657 0j	.37353 0j	.083931 0j	.23290 0j	.37353 0j	.28657 0j	.08393 0j
.25	.28657 -.09357j	.36813 -.18833j	.083499 -.018750j	.23149 -.07199j	.37164 -.03124j	.27687 -.18605j	.06516 -.11702j
.50	.28657 -.18713j	.33038 -.38354j	.082294 -.037763j	.22755 -.14484j	.36637 -.06362j	.24759 -.37156j	.00856 -.23384j
.83	.28657 -.31188j	.26422 -.66055j	.079784 -.063745j	.21974 -.24404j	.35540 -.10956j	.17713 -.61364j	-.12688 -.38333j
1.25	.28657 -.46783j	.15577 -1.03588j	.075671 -.097328j	.20590 -.37165j	.33741 -.17181j	.03948 -.90545j	-.39549 -.56127j
1.67	.28657 -.62377j	.03007 -1.14273j	.070904 -.13216[illegible]j	.19030 -.50317j	.31656 -.23930j	-.16983 -1.18273j	-.77815 -.78660j

e = .2 VI-25

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8860 -3.1860j	-1.39289 -1.14626j	-3.69996 -.829573j	.10206 -.09071j	-.10014 -.46873j	-.29651 -.30652j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-2.23498 -1.43283j	-6.05566 -.67732j	.09361 -.11853j	-.22347 -.56998j	-.49037 -.37450j
2.94	-.0820 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-3.13340 -1.68568j	-8.66488 -.41580j	.08614 -.14434j	-.35912 -.65381j	-.70075 -.42614j
3.33	-.1990 -4.4333j	.37500 -3.33333j	-15.4730 -3.7522j	-4.05429 -1.91041j	-11.41763 -.09971j	.07957 -.16116j	-.50171 -.72326j	-.91540 -.46849j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-5.15888 -2.14924j	-14.76593 .36620j	.07256 -.19376j	-.67470 -.79203j	-1.18345 -.50933j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-6.39343 -2.38806j	-18.66031 .88187j	.06603 -.22091j	-.87517 -.85847j	-1.48432 -.54791j

v/bω	M_ζ	L_ζ	T_z	P_z	P_h	P_α	P_β
2.00	.28657 -.74852j	-.07724 -1.78841j	.06683 -.16070j	.17699 -.61089j	.29877 -.29663j	-.38146 -1.39316j	-1.16361 -.64918j
2.50	.28657 -.93565j	-.24340 -2.33684j	.06053 -.20471j	.15639 -.77518j	.27121 -.38759j	-.78473 -1.68937j	-1.89863 -1.01688j
2.94	.28657 -1.10077j	-.39071 -2.84556j	.05494 -.24449j	.13811 -.92510j	.24677 -.47197j	-1.22833 -1.93146j	-2.68912 -1.14906j
3.35	.28657 -1.24753j	-.52008 -3.31520j	.05004 -.28051j	.12207 -1.05963j	.22531 -.54987j	-1.69457 -2.13244j	-3.50917 -1.25493j
3.75	.28657 -1.40346j	-.65829 -3.81998j	.04479 -.31901j	.10492 -1.20330j	.20239 -.63359j	-2.26021 -2.32826j	-4.49866 -1.35387j
4.17	.28657 -1.55942j	-.78724 -4.35525j	.03991 -.35866j	.08895 -1.35075j	.18104 -.72237j	-2.91576 -2.51644j	-5.52338 -1.44539j

e =.2

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-9.25235 -2.86565j	-27.85571 2.19023j	.05336 -.27598j	-1.34728 -.97[illegible]55j	-2.18778 -.61456j
6.25	-1.3450 -9.5350j	.37500 -6.25000j	-61.4370 -1.1288j	-14.51538 -3.58206j	-45.40227 4.70193j	.03595 -.36166j	-2.24512 -1.12994j	-3.50625 -.69368j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-25.88409 -4.77606j	-84.85530 10.06292j	.01103 -.50972j	-4.25750 -1.52653j	-6.41332 -.78101j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-37.32185 -5.73130j	-125.72525 15.18685j	-.00581 -.63116j	-6.33811 -1.44279j	-9.38226 -.81921j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-58.37398 -7.16413j	-202.60578 23.96334j	-.02720 -.81587j	-10.24733 -1.56301j	14.91006 -.83512j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-103.86116 -9.55219j	-372.43549 40.78198j	-.05538 -1.12779j	-18.37424 -1.65406j	-26.99437 -.77855j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.28657 -1.8713j	-1.0366 -5.4409j	.03044 -.43853j	.05798 -1.64753j	.13961 -.90244j	-4.4596 -2.8480j	-8.2499 -1.5941j
6.25	.28657 -2.39913j	-1.3800 -7.1302j	.01742 -.56064j	.01541 -2.10022j	.08268 -1.18263j	-7.3955 -3.2588j	-13.1619 -1.7490j
8.33	.28657 -3.11883j	-1.8710 -10.0492j	-.00122 -.76809j	-.04553 -2.86754j	.00119 -1.66678j	-13.9759 -3.7562j	-23.3656 -[illegible]8614j
10.00	.28657 -3.7426j	-2.2034 -12.4432j	-.01381 -.93628j	-.08671 -3.46570j	-.05388 -2.06386j	-20.7794 -4.0201j	-34.9790 -1.8475j
12.50	.28657 -4.67825j	-2.6251 -16.0850j	-.02481 -1.19049j	-.13902 -4.42673j	-.12386 -2.66789j	-33.5625 -4.2388j	-55.4572 -1.6921j
16.67	.28657 -6.[illegible]3766j	-3.1807 -22.2343j	-.05088 -1.62719j	-.20793 -6.00001j	-.21599 -3.68783j	-61.7722 -4.2457j	-100.1630 -1.1609j

e = .3

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_ξ
0	1.0000 0j	.37500 0j	.50000 0j	.07740 0j	.09064 0j	.090640 0j	.077402 0j	.020510 0j
.25	.9845 -.2519j	.37500 -.25000j	.42179 -.49423j	.05273 -.12351j	.04098 -.15789j	.090232 -.006747j	.075306 -.045431j	.016529 -.028553j
.50	.9425 -.5129j	.37500 -.50000j	.18580 -.98405j	-.02128 -.24701j	-.11016 -.30929j	.089094 -.013741j	.068984 -.090743j	.005125 -.056952j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.59487j	-.19670 -.41167j	-.47905 -.47304j	.086725 -.023664j	.053765 -.150020j	-.022188 -.093748j
1.25	.7088 -1.3855j	.37500 -1.25000j	-1.52280 -2.27119j	-.53938 -.61753j	-1.23264 -.60181j	.082839 -.037111j	.023211 -.221800j	-.076395 -.157756j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-1.02914 -.82358j	-2.33726 -.63236j	.078336 -.051687j	-.021049 -.290430j	-.153683 -.179020j

v/b ω	M_ξ	L_ξ	T_ξ	P_ξ	P_h	P_α	P_β
0	.24701 0j	.31192 0j	.05700 0j	.18120 0j	.31192 0j	.24701 0j	.05700 0j
.25	.24701 -.09869j	.30114 -.17797j	.05671 -.01457j	.18013 -.06379j	.31041 -.02503j	.23924 -.15539j	.04054 -.09079j
.50	.24701 -.19737j	.27115 -.36245j	.05591 -.02932j	.17715 -.12822j	.30619 -.05096j	.21577 -.31034j	-.00905 -.18094j
.83	.24701 -.32895j	.20861 -.62421j	.05423 -.04940j	.17093 -.21568j	.29739 -.02777j	.15534 -.51273j	-.12766 -.29734j
1.25	.24701 -.49343j	.10614 -.97890j	.05349 -.07525j	.16475 -.32778j	.28279 -.13764j	.04602 -.75705j	-.36252 -.45531j
1.67	.24701 -.65790j	-.01265 -1.36338j	.04831 -.10189j	.14895 -.44223j	.26626 -.19171j	-.11814 -.98976j	-.69650 -.56351j

21516

e=.3

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3772 -2.3916j	.37500 -2.00000j	-4.8850 -3.1560j	-1.50156 -.96804j	-3.49362 -.58192j	.074491 -.064071j	-.06689 -.34287j	-.23299 -.21017j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-2.38970 -1.23505j	-5.70957 -.38075j	.068538 -.083719j	-.15400 -.41735j	-.38022 -.25343j
2.94	-.0220 -3.3053j	.37500 -2.94118j	-11.7140 -3.7396j	-3.33731 -1.45300j	-8.16445 -.08139j	.063261 -.101944j	-.24981 -.47889j	-.54011 -.28833j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7827j	-4.30860 -1.64657j	-10.75713 .27435j	.058626 -.118769j	-.35051 -.53053j	-.70607 -.31695j
3.75	-.3798 -5.1064j	.37500 -3.75000j	-20.0337 -3.6847j	-5.47363 -1.85258j	-13.91200 .75614j	.053674 -.136653j	-.47269 -.58156j	-.90635 -.34458j
4.17	-.5520 -5.5242j	.37500 -4.16667j	-25.3190 -3.5256j	-6.77574 -2.05843j	-17.58354 1.29644j	.049062 -.156029j	-.61429 -.63095j	-1.13408 -.37065j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.24701 -.78948j	-.11405 -1.69005j	.04559 -.12372j	.13887 -.55374j	.25302 -.23764j	-.28817 -1.16679j	-1.03677 -.65864j
2.50	.24701 -.96685j	-.27107 -2.20831j	.04158 -.15721j	.12527 -.69044j	.22995 -.31653j	-.61125 -1.41675j	-1.67186 -.78885j
2.94	.24701 -1.16100j	-.41029 -2.68905j	.03765 -.18739j	.10944 -.80956j	.21037 -.37811j	-.96662 -1.62159j	-2.35846 -.89164j
3.33	.24701 -1.32580j	-.53254 -3.13286j	.03438 -.21466j	.09729 -.92597j	.19318 -.44051j	-1.34013 -1.79253j	-3.05978 -.97410j
3.75	.24701 -1.48028j	-.66315 -3.60937j	.03083 -.24378j	.08431 -1.05024j	.17482 -.50759j	-1.79331 -1.96027j	-3.92756 -1.05139j
4.17	.24701 -1.64475j	-.78482 -4.11570j	.02762 -.27366j	.07222 -1.17730j	.15771 -.57571j	-2.31847 -2.12158j	-4.90066 -1.12288j

21516

e = .3

$v/b\omega$	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-9.79110 -2.47010j	-26.2564 2.6398j	.040114 -.194924j	-.94774 -.72004j	-1.66603 -.41574j
6.25	-1.3450 -9.5350j	.37500 -6.2500j	-61.4370 -1.1288j	-15.34213 -3.08763j	-42.8149 5.1780j	.027818 -.255443j	-1.58189 -.83499j	-2.66116 -.46935j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-27.33260 -4.11682j	-80.0650 10.5253j	.010217 -.360017j	-3.00323 -.98615j	-4.85065 -.52878j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-39.39660 -4.94020j	-118.6647 15.5963j	-.001678 -.445785j	-4.47275 -1.07811j	-7.08315 -.55508j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-61.66060 -6.17525j	-191.2884 24.2369j	-.016788 -.576253j	-7.23386 -1.17778j	-11.23487 -.56683j
16.67	-3.7530 -89.7333j	.37500 -16.66667j	-499.8530 32.8222j	-109.57660 -8.23368j	-351.7399 40.7138j	-.036693 -.796556j	-13.32705 -1.26671j	-20.29972 -.53079j

$v/b\omega$	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.84701 -1.97370j	-1.0208 -5.1417j	.02150 -.33384j	.04877 -1.43293j	.12452 -.72297j	-3.5553 -2.4083j	-7.1711 -1.2397j
6.25	.24701 -2.46713j	-1.3452 -6.7380j	.01261 -.42563j	.01654 -1.82205j	.07891 [illegible]	-5.9073 -2.7691j	-11.4096 -1.3630j
8.33	.24701 -3.28950j	-1.8095 -9.4965j	.00017 -.58124j	-.02959 -2.48031j	.01383 -1.33530j	-11.1791 -3.2204j	-20.7434 -1.4576j
10.00	.24701 -3.94740j	-2.1232 -11.7568j	-.00824 -.70721j	-.06077 -3.01245j	-.03049 -1.65842j	-16.6236 -3.4740j	-30.1535 -1.4544j
12.50	.24701 -4.93425j	-2.5218 -15.2003j	-.01891 -.89746j	-.10037 -3.81542j	-.08653 -2.13732j	-25.8704 -3.7126j	-47.7725 -1.3466j
16.67	.24701 -6.57900j	-3.0458 -21.0114j	-.03298 -1.21655j	-.15254 -5.16120j	-.16036 -2.95442j	-49.4701 -3.8238j	-86.1262 -.9613j

21516

e = .4

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.05775 0j	.06238 0j	.062380 0j	.057752 0j	.011070 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	.03222 -.10392j	.01666 -.12611j	.052106 -.004523j	.056346 -.031104j	.008275 -.018213j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	-.04436 -.20785j	-.12270 -.24573j	.061343 -.009222j	.052103 -.062128j	-.000147 -.036248j
.83	.8538 -.8533j	.37500 -.83333j	-.38230 -1.59487j	-.22588 -.34639j	-.46352 -.36854j	.059751 -.015852j	.041888 -.102730j	-.020303 -.059676j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.58044 -.51961j	-1.16048 -.49926j	.057446 -.024907j	.021382 -.151920j	-.060230 -.087653j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-1.07681 -.69283j	-2.18647 -.43617j	.054122 -.034689j	-.008322 -.179000j	-.117034 -.113943j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.20785 0j	.25231 0j	.036367 0j	.13515 0j	.25231 0j	.20785 0j	.03637 0j
.25	.20785 -.10211j	.24223 -.16641j	.036186 -.010882j	.13437 -.05541j	.25113 -.01946j	.20181 -.12571j	.02219 -.06757j
.50	.20785 -.20422j	.21419 -.33890j	.035662 -.021873j	.13221 -.11130j	.24785 -.03963j	.18357 -.25108j	-.02049 -.13465j
.83	.20785 -.34037j	.15572 -.58366j	.034630 -.036793j	.12769 [illegible]	.24101 -.06826j	.13967 -.41196j	-.12248 -.22134j
1.25	.20785 -.51055j	.05990 -.91530j	.032908 -.055907j	.12028 -.28551j	.22981 -.10704j	.05154 -.61311j	-.32410 -.32386j
1.67	.20785 -.63073j	-.05117 -1.27480j	.030911 -.075521j	.11170 -.38222j	.21682 -.11909j	-.076121 -.80220j	-.61026 -.41923j

21516

$e = .4$

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8860 -3.1860j	-1.57601 -.83138j	-3.26105 -.35445j	.051542 -.043001j	-.03909 -.23502j	-.17493 -.13369j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-2.49500 -1.03923j	-5.32272 -.11264j	.047547 -.056187j	-.09755 -.28623j	-.28300 -.16118j
2.94	-.0220 -3.8053j	.37500 -2.94112j	-11.7140 -3.7396j	-3.47548 -1.22262j	-7.60897 .21620j	.044005 -.068419j	-.16185 -.32863j	-.39987 -.18335j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-4.48047 -1.38562j	-10.02518 .59342j	.040894 -.079711j	-.22944 -.36423j	-.52097 -.20153j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-5.68594 -1.55884j	-12.96651 1.09162j	.037571 -.091848j	-.31145 -.39952j	-.66700 -.21910j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-7.03322 -1.73206j	-16.3153 1.64643j	.034475 -.104718j	-.40647 -.43386j	-.83269 -.23565j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.20785 -.81688j	-.14599 -1.58025j	.02921 -.09153j	.10437 -.46255j	.20573 -.18481j	-.20834 -.94637j	-.90132 -.49002j
2.50	.20785 -1.02110j	-.29281 -2.06484j	.02657 -.11603j	.09303 -.58511j	.18856 -.24148j	-.45960 -1.15050j	-1.44364 -.58699j
2.94	.20785 -1.20130j	-.42298 -2.52435j	.02423 -.13803j	.08297 -.69494j	.17334 -.29405j	-.73596 -1.31864j	-2.02891 -.66363j
3.33	.20785 -1.36147j	-.53729 -2.92932j	.02217 -.15787j	.07414 -.79375j	.15997 -.34258j	-1.02643 -1.45922j	-2.63443 -.72520j
3.75	.20785 -1.53165j	-.65941 -3.37535j	.01998 [illegible]	.06470 -.89915j	.14569 -.39474j	-1.37835 -1.59755j	-3.36413 -.75308j
4.17	.20785 -1.70183j	-.77317 -3.84531j	.017[illegible]3 -.2006[illegible]j	.05591 -1.00662j	.13238 -.45005j	-1.78726 -1.73113j	-4.19029 -.83659j

e = .4

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-10.1533 -2.0785j	-24.4855 3.0027j	.028470 -.130822j	-.63027 -.49552j	-1.21930 -.26431j
6.25	-1.3450 -9.5350j	.37500 -6.250000j	-61.4370 -1.1288j	-15.9869 -2.5981j	-39.9469 5.5299j	.020217 -.171439j	-1.05587 -.57573j	-1.94111 -.29845j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-28.3039 -3.4641j	-74.7464 10.7930j	.008404 -.241642j	-2.00980 -.68229j	-3.52564 -.33645j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-40.7863 -4.1569j	-110.8178 15.7487j	.000421 -.299187j	-2.99610 -.74810j	-5.13856 -.35345j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-63.7610 -5.1961j	-178.6972 24.1523j	-.009719 -.386750j	-4.84917 -.82113j	-8.13428 -.36150j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-113.3978 -6.9282j	-328.6900 40.1043j	-.023079 -.534605j	-8.93860 -.89102j	-14.66661 -.33994j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.20785 -2.04220j	-.9939 -4.8076j	.01396 -.24424j	.03886 -1.22260j	.10657 -.56224j	-2.7491 -1.9704j	-6.1160 -.9245j
6.25	.20785 -2.552755j	-1.2972 -6.3003j	.00851 -.31053j	.01542 -1.55072j	.07111 -.73680j	-4.5782 -2.2753j	-9.7045 -1.0184j
8.33	.20785 -3.40367j	-1.7313 -8.8795j	.00070 -.42265j	-.01813 -2.10463j	.02034 -1.03843j	-8.6779 -2.6669j	-17.5651 -1.0936j
10.00	.20785 -4.08440j	-2.0246 -10.9349j	-.00457 -.51328j	-.04080 -2.55181j	-.01397 -1.28562j	-12.9167 -2.8966j	-25.5537 -1.0962j
12.50	.20785 -5.10550j	-2.3973 -14.2127j	-.01127 -.65004j	-.06959 -3.22601j	-.05756 -1.66214j	-20.8807 -3.1308j	[illegible] -1.0251j
16.67	.20785 -6.80733j	-2.8882 -19.6463j	-.02010 -.87923j	-.10755 -4.35514j	-.11437 -2.29758j	[illegible] -3.2985j	-72.6508 -.7563j

e =.5 VI-33

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.03587 0j	.04008 0j	.04008 0j	.035870 0j	.005390 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	.01003 -.05333j	-.00142 -.09630j	.039909 -.002833j	.034990 -.019777j	.003434 -.010639j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	-.06751 -.16666j	-.12306 -.13622j	.039431 -.005770j	.032335 -.039905j	-.002461 -.021205j
.83	.8538 -.8833j	.37500 -.83333j	-.38230 -1.99487j	-.25126 -.27776j	-.43843 -.27137j	.038435 -.009937j	.025944 -.066000j	-.016548 -.034904j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.61022 -.41665j	-1.07467 -.30907j	.036804 -.015584j	.013113 -.037638j	-.044401 -.051253j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-1.11274 -.55554j	-2.01318 -.25851j	.034913 -.021705j	-.005472 -.12796j	-.083934 -.066609j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.16666 0j	.19550 0j	.021310 0j	.09521 0j	.19550 0j	.16666 0j	.02131 0j
.25	.16666 -.10338j	.18621 -.15338j	.021206 -.007685j	.09468 -.04684j	.19462 -.01452j	.16216 -.09742j	.00942 -.04752j
.50	.16666 -.20675j	.16036 -.31236j	.020915 -.015434j	.09329 -.09400j	.19217 -.02958j	.14855 -.19457j	-.02638 -.09468j
.83	.16666 -.34458j	.10846 -.53795j	.020308 -.029918j	.09008 -.15767j	.18707 -.05094j	.13579 -.32170j	-.11183 -.15555j
1.25	.16666 -.51688j	.01816 -.84368j	.019515 -.039290j	.08499 -.23863j	.17871 -.07989j	.05002 -.47556j	-.29045 -.22767j
1.67	.16666 -.68917j	-.08421 -1.17496j	.018163 -.052952j	.07908 -.32107j	.16901 -.11126j	-.04526 -.62267j	-.51929 -.29470j

21516

e = .5 VI-34

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.30161j	.37500 -2.00000j	-4.8860 -3.1860j	-1.61813 -.66664j	-2.99752 [illegible]	.033299 -.026905j	-.02472 -.15115j	-.12414 [illegible]
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-2.54851 -.83330j	-4.88830 .12122j	.030799 -.035156j	-.06130 -.18425j	-.19905 -.09419j
2.94	-.0220 -3.5053j	.37500 -2.94118j	-11.7140 -3.7396j	-3.54313 -.980355j	-6.98715 .46947j	.028583 -.042809j	-.10154 -.21169j	-.27990 -.10723j
3.33	-.1959 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-4.55857 -1.11106j	-9.20679 .85827j	.026636 -.049875j	-.14382 -.23478j	-.36352 -.11775j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-5.77897 -1.24995j	-11.90993 1.36150j	.024557 -.057469j	-.19513 -.25772j	-.46430 -.12801j
4.17	-.5529 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-7.14296 -1.38834j	-15.05940 1.91862j	.022620 -.065521j	-.25459 -.27996j	-.57837 -.13767j

v/bω	M_z	L_z	T_z	P_z	R_z	P_α	P_β
2.00	.16666 -.82700j	-.17161 -1.45648j	.01718 -.06407j	.07404 -.38795j	.16074 -.13798j	-.14394 -.73507j	-.76179 -.34448j
2.50	.16666 -1.033375j	-.30693 -1.90313j	.01566 -.08101j	.06624 -.48971j	.14792 -.18021j	-.33144 -.89460j	-1.21279 -.41270j
2.94	.16666 -1.21618j	-.42690 -2.31742j	.01431 -.09619j	.05932 -.58065j	.13656 -.21944j	-.55769 -1.02643j	-1.69866 -.46669j
3.33	.16666 -1.37833j	-.53226 -2.69990j	.01312 -.10984j	.05325 -.66230j	.12659 -.25566j	-.75447 -1.13699j	-2.20060 -.51010j
3.75	.16666 -1.55063j	-.64482 -3.11099j	.01186 -.12440j	.04675 -.74934j	.11593 -.29459j	-1.01748 -1.24623j	-2.80507 -.55103j
4.17	.16666 -1.72291j	-.74967 -3.54691j	.01068 -.13924j	.04071 -.83780j	.10600 -.33587j	-1.32226 -1.35192j	-3.46805 -.58885j

21516

$e = .5$

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-37.7660 -2.8460j	-10.3016 -1.6666j	-22.5053 3.2611j	.018863 -.081855j	-.39463 -.32037j	-.84425 -.15442j
6.25	-1.3450 -9.5350j	.37500 -6.25000j	-61.4370 -1.1288j	-16.1165 -2.0833j	-36.7363 5.7324j	.013699 -.107269j	-.66092 -.37314j	-1.33959 -.17459j
8.33	-2.0080 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-26.6772 -2.7777j	-68.7825 10.8258j	.006308 -.151183j	-1.25779 -.44410j	-2.42440 -.19671j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-41.3141 -3.3332j	-102.0100 15.5908j	.001313 -.187200j	-1.87490 -.48872j	-3.52668 -.20681j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-64.5735 -4.1665j	-164.5231 23.6352j	-.005032 -.241987j	-3.03437 -.53958j	-5.57124 -.21186j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-114.8252 -5.5553j	-302.7630 58.8406j	-.013391 -.334500j	-5.59310 -.59192j	-10.02340 -.20005j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.16666 -2.06750j	-.9531 -4.4311j	.00839 -.16905j	.02898 -1.01544j	.08674 -.41959j	-2.0401 -1.5425j	-5.0784 -.6513j
6.25	.16666 -2.58438j	-1.2326 -5.8068j	.00524 -.21432j	.01286 -1.28476j	.06027 -.54986j	-3.4051 -1.7880j	-8.0363 -.7187j
8.33	.16666 -3.44583j	-1.6327 -8.1841j	.00074 -.29073j	-.01022 -1.73847j	.02238 -.77497j	-6.4647 -2.1103j	-14.5018 -.7747j
10.00	.16666 -4.13900j	-1.9031 -10.1338j	-.00830 -.35240j	-.02581 -2.10422j	-.00322 -.95960j	-9.6280 -2.3057j	-21.0620 -.7797j
12.50	.16666 -5.16875j	-2.2466 -13.0996j	-.00616 -.44537j	-.04568 -2.65522j	-.03575 -1.24044j	-15.5716 -2.5165j	-33.2175 -.7355j
16.67	.16666 -6.99166j	-2.6991 -18.1076j	-.01125 -.60104j	-.07172 -3.57728j	-.07860 -1.71465j	-28.5678 -2.7016j	-59.6582 -.5561j

21516

e = .6

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.02126 0j	.02322 0j	.023220 0j	.021260 0j	.002230 0j
.25	.9848 -.2519j	.37500 -.25000j	.42179 -.49423j	-.00420 -.06276j	-.01367 -.06886j	.023123 -.001602j	.020763 -.011574j	.000964 -.005505j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	-.08060 -.12552j	-.12639 -.13162j	.022853 -.003262j	.019262 -.023119j	-.002850 -.010973j
.83	.5538 -.8833j	.37500 -.83333j	-.38230 -1.59487j	-.26166 -.20919j	-.40325 -.18305j	.022290 -.005618j	.015648 -.038243j	-.011947 -.018058j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.61537 -.31380j	-.97216 -.18385j	.021368 -.008810j	.008395 -.056595j	-.029897 -.026509j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-1.11052 -.41841j	-1.81310 -.10336j	.020299 -.012270j	-.002112 -.074201j	-.055311 -.034444j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.12552 0j	.14238 0j	.011042 0j	.06177 0j	.14238 0j	.12552 0j	.01104 0j
.25	.12552 -.10186j	.13400 -.13847j	.010989 -.005008j	.06138 -.03803j	.14176 -.01021j	.12235 -.07096j	.00144 -.03079j
.50	.12552 -.20372j	.11066 -.28200j	.010840 -.010049j	.06049 -.07627j	.14004 -.02078j	.11273 -.14173j	-.02744 -.06134j
.83	.12552 -.33953j	.06200 -.48567j	.010551 -.016847j	.05852 -.12775j	.13646 -.03579j	.08977 -.23439j	-.09630 -.10076j
1.25	.12552 -.50930j	-.0177 -.76164j	.010024 -.025481j	.05529 -.19297j	.13658 -.05613j	.04355 -.34668j	-.23193 -.14746j
1.67	.12552 -.67907j	-.11015 -1.06078j	.009436 -.0342[illegible]j	.05154 -.25914j	.12377 -.07818j	-.02539 -.45423j	-.42363 -.19087j

e = .6 VI-37

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916j	.37500 -2.00000j	-4.8860 -3.1860j	-1.60850 -.50208j	-2.69634 .02237j	.019386 -.015211j	-.012995 -.087703j	-.081106 -.040401j
2.50	.1752 -3.1250j	.37500 -2.50000j	-8.1375 -3.5625j	-2.52524 -.62760j	-4.39496 .31324j	.017973 -.019875j	-.033675 -.106960j	-.129060 -.048688j
2.94	-.0220 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-3.50332 -.73835j	-6.28238 .66849j	.016720 -.024202j	-.056421 -.122960j	-.180690 -.055372j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-4.50585 -.83679j	-8.27476 1.05671j	.015620 -.028196j	-.080326 -.136460j	-.234010 -.060853j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-5.70837 -.94140j	-10.71322 1.55085j	.014444 -.032489j	-.109340 -.149830j	-.298230 -.066206j
4.17	-.5520 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-7.05236 -1.04601j	-13.55010 2.09519j	.013349 -.037042j	-.142950 -.163920j	-.370730 -.071142j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.12552 -.81488j	-.18905 -1.31495j	.008934 -.041383j	.04834 -.31266j	.11795 -.09691j	-.09273 -.53654j	-.61792 -.22311j
2.50	.12552 -1.01860j	-.31122 -1.71819j	.008157 -.052203j	.04339 -.39383j	.10895 -.12663j	-.22448 -.65366j	-.97855 -.26732j
2.94	.12552 -1.19835j	-.41954 -2.[illegible]j	.007468 -.061866j	.03900 -.46622j	.10097 -.15429j	-.36940 -.75070j	-1.36637 -.30234j
3.33	.12552 -1.35813j	-.51465 -2.43753j	.006863 -.070536j	.03515 -.53107j	.09396 -.17964j	-.52172 -.83232j	-1.75638 -.33054j
3.75	.12552 -1.52790j	-.61688 -2.80868j	.006217 -.079776j	.03103 -.60015j	.08647 -.20700j	-.70652 -.91325j	-2.24779 -.35718j
4.17	.12552 -1.69767j	-.71094 -3.20224j	.005615 -.089159j	.02720 -.67014j	.07949 -.23600j	-.92069 -.97169j	-2.79054 -.38180j

21516

$e = .6$

v/b ω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.8860 -7.2760j	.37500 -5.00000j	-57.7660 -2.8460j	-10.1647 -1.2552j	-20.2598 3.3907j	.011225 -.046275j	-.22212 -.18670j	-.53952 -.07980j
6.25	-1.3450 -9.5350j	.37500 -6.2500j	-61.4370 -1.1288j	-15.8944 -1.5690j	-33.0905 5.7502j	.008306 -.060643j	-.37266 -.21793j	-.85329 -.09013j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-28.2710 -2.0920j	-61.3976 10.5678j	.004127 -.085469j	-.71009 -.26038j	-1.53874 -.10172j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.3460 7.8200j	-40.7227 -2.5104j	-91.9794 15.0480j	.001304 -.105330j	-1.05896 -.28746j	-2.23391 -.10702j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-63.6412 -3.1380j	-148.4198 22.5806j	-.002284 -.136804j	-1.71444 -.31901j	-[illegible].52155 -.10979j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-113.0432 -4.1840j	-273.1765 36.7624j	-.007009 -.189104j	-3.16099 -.35325j	-6.32144 -.10406j

v/b ω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
5.00	.12552 -2.03720j	-.8946 -4.0005j	.004447 -.107994j	.01976 -.81056j	.06596 -.29483j	-1.4251 -1.1339j	-4.05312 -.42264j
6.25	.12552 -2.54650j	-1.1470 -5.2425j	.002842 -.136531j	.00953 -1.02300j	.04736 -.38637j	-2.3842 -1.3190j	-6.39680 -.46704j
8.33	.12552 -3.39533j	-1.5082 -7.3888j	.000544 -.184582j	-.00511 -1.38015j	.02074 -.54454j	-4.5340 -1.5663j	-11.50834 -.50518j
10.00	.12552 -4.0714j	-1.7523 -9.1490j	-.001009 -.223290j	-.01500 -1.66763j	.00275 -.67427j	-6.7568 -1.7203j	-16.68607 -.51031j
12.50	.12552 -5.09300j	-2.0624 -11.8266j	-.002981 -.281545j	-.02757 -2.10036j	-.02011 -.87160j	-10.9330 -1.8935j	-26.26820 -.48510j
16.67	.12552 -6.79066j	-2.4709 -16.3480j	-.005579 -.379144j	-.04412 -2.82395j	-.05022 -1.20482j	-20.1492 -2.0653j	-47.08457 -.37714j

e =.7

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
0	1.0000 0j	.37500 0j	.50000 0j	.01072 0j	.91145 0j	.011450 0j	.010720 0j	.000710 0j
.25	.9846 -.2519j	.37500 -.25000j	.42179 -.49423j	-.01343 -.04284j	-.02028 -.04431j	.011403 -.000771j	.010480 -.005706j	-.000012 -.002348j
.50	.9423 -.5129j	.37500 -.50000j	.18580 -.98405j	-.08589 -.08567j	-.11740 -.08298j	.011274 -.001569j	.009758 -.011398j	-.002182 -.004673j
.83	.8538 -.8833j	.37500 -.83333j	-.[illegible]3230 -1.59487j	-.25762 -.14278j	-.35645 -.10565j	.011003 -.002703j	.008020 -.018857j	-.007352 -.007697j
1.25	.7088 -1.3853j	.37500 -1.25000j	-1.52280 -2.27119j	-.59309 -.21418j	-.84886 -.07719j	.010559 -.004239j	.004579 -.027916j	-.017532 -.011296j
1.67	.5407 -1.9293j	.37500 -1.66667j	-3.17490 -2.83045j	-1.06[illegible]72 -.28557j	-1.57826 .02370j	.010045 -.005904j	-.000528 -.0366[illegible]j	-.031905 -.014672j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
0	.08567 0j	.09406 0j	.004713 0j	.03520 0j	.09406 0j	.08567 0j	.00471 0j
.25	.08567 -.09661j	.08673 -.12101j	.004689 -.002864j	.03501 -.02897j	.09367 -.00651j	.08365 -.04688j	-.00257 -.01753j
.50	.08567 -.19322j	.06634 -.24645j	.004627 -.005749j	.03448 -.05805j	.09257 -.01327j	.07754 -.09365j	-.02446 -.03491j
.83	.08567 -.3[illegible]203j	.02381 -.42444j	.004496 -.009623j	.03338 -.09710j	.09028 -.02285j	.06285 -.15492j	-.07657 -.05734j
1.25	.08567 -.48305j	-.04586 -.66561j	.004283 -.014523j	.03158 -.14[illegible]j	.08653 -.03583j	.05335 -.22924j	-.17902 -.08392j
1.67	.08567 -.64407j	-.12663 -.92704j	.004036 -.019486j	.02949 -.19623j	.08218 -.04990j	-.00938 -.30055j	-.32350 -.10861j

2/5/6

C = .7

VI-40

v/bω	M_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
2.00	.3972 -2.3916 j	.37500 -2.00000j	-4.8860 -3.1860j	-1.53504 -.34268j	-2.34542 .15890j	.009606 -.007318j	-.005764 -.043295j	-.046462 -.017208j
2.50	.1752 -3.1250j	.37500 -2.500000j	-8.1375 -3.5625j	-2.40453 -.42835j	-3.82258 .45238j	.008926 -.009562j	-.015716 -.052833j	-.073460 -.020734j
2.94	-.0220 -3.8053j	.37500 -2.94118j	-11.7140 -3.7396j	-3.33219 -.50394j	-5.46557 .79861j	.008323 -.011644j	-.026662 -.060773j	-.102470 -.023578j
3.33	-.1950 -4.4333j	.37500 -3.33333j	-15.4730 -3.7822j	-4.28306 -.57108j	-7.20544 1.17042j	.007793 -.013566j	-.038166 -.067481j	-.132370 -.025920j
3.75	-.3798 -5.1084j	.37500 -3.75000j	-20.0337 -3.6847j	-5.42359 -.64253j	-9.32601 1.63711j	.007228 -.015632j	-.052124 -.074171j	-.168350 -.028168j
4.17	-.5526 -5.8242j	.37500 -4.16667j	-25.3190 -3.5256j	-6.69832 -.71392j	-11.79957 2.24900j	.006701 -.017822j	-.068300 -.080673j	-.208870 -.030288j

v/bω	M_z	L_z	T_z	P_z	P_h	P_α	P_β
2.00	.08567 -.77288j	-.19559 -1.14916j	.003825 -.025496j	.02771 -.23642j	.07847 -.06186j	-.05364 -.35520j	-.46964 -.12696j
2.50	.08567 -.95620j	-.30235 -1.50156j	.003498 -.029569j	.02495 -.29721j	.07272 -.08033j	-.13775 -.43313j	-.74039 -.15214j
2.94	.08567 -1.13659j	-.39701 -1.82244j	.003209 -.034976j	.02250 -.35126j	.06765 -.09843j	-.23026 -.49789j	-1.03098 -.17210j
3.33	.08567 -1.28813j	-.48014 -2.13022j	.002954 -.039327j	.02035 -.39959j	.06315 -.11467j	-.32749 -.55248j	-1.33025 -.18818j
3.75	.08567 -1.44915j	-.56895 -2.45457j	.002683 -.044971j	.01805 -.45105j	.05849 -.13213j	-.44545 -.60679j	-1.59015 -.20341j
4.17	.08567 -1.61017j	-.65168 -2.79851j	.002430 -.050176j	.01591 -.50301j	.05392 -.15065j	-.58216 -.65950j	-2.09503 -.21748j

2 '8/5

e = .7

v/bω	L_h	M_α	L_α	M_β	L_β	T_h	T_α	T_β
5.00	-.5860 -7.2760j	.37500 -5.00000j	-37.7660 -2.[illegible]60j	-9.6503 -.8567j	-17.6323 3.3542j	.005679 -.022265j	-.10035 -.09257j	-.30309 -.03397j
6.25	-1.3450 -9.5350j	.37500 -6.25000j	-61.4370 -1.1288j	-15.3020 -1.0709j	[illegible] 5.5284j	.004274 -.029177j	-.17884 -.10825j	-.47784 -.03838j
8.33	-2.0020 -13.4385j	.37500 -8.33333j	-114.4920 3.2420j	-26.8233 -1.4278j	-54.0913 9.9302j	.002264 -.041122j	-.34122 -.12964j	-.85859 -.04334j
10.00	-2.4460 -16.6400j	.37500 -10.00000j	-169.[illegible] 7.8200j	-38.6333 -1.7134j	-80.2784 14.0014j	.000905 -.050718j	-.50911 -.14378j	-1.24398 -.04563j
12.50	-3.0100 -21.5100j	.37500 -12.50000j	-272.4100 16.1150j	-60.3793 -2.1415j	-129.5845 20.5203j	-.000821 -.065821j	-.82453 -.16032j	-1.95679 -.04688j
16.67	-3.7530 -29.7333j	.37500 -16.66667j	-499.8530 32.8222j	-107.3327 -2.8557j	-238.5860 33.6109j	-.003094 -.090784j	-1.52084 -.17308j	-3.50441 -.04461j

v/bω	M_z	L_z	T_z	R_z	I_h	P_α	P_β
5.00	.08567 -1.93220j	-.8122 -3.4961j	.001939 -.060646j	.01176 -.60717j	.04528 -.18820j	-.90411 -.75558j	-3.03588 -.24093j
6.25	.08567 -2.41525j	-1.0327 -4.5816j	.001264 -.076453j	.00606 -.7644j	.03341 -.24663j	-1.51639 -.88166j	-4.77886 -.26660j
8.33	.08567 -3.22033j	-1.3444 -6.4572j	.000298 -.103005j	-.00211 -1.02225j	.01641 -.34760j	-2.99270 -1.05277j	-8.57160 -.28928j
10.00	.08567 -3.86440j	-1.5617 -7.9955j	-.000355 -.124361j	-.00763 -1.24029j	.00493 -.43044j	-4.30755 -1.16167j	-12.40686 -.29320j
12.50	.08567 -4.83050j	-1.8328 -10.3356j	-.001184 -.156495j	-.01464 -1.55919j	-.00966 -.55632j	-6.97330 -1.28810j	-19.49554 -.28066j
16.67	.08567 -6.44066j	-2.1898 -14.2869j	-.002276 -.210208j	-.02327 -2.09201j	-.02888 -.75608j	-12.85641 -1.14426j	-34.87473 -.22279j

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