REPORT No. 736

NONSTATIONARY FLOW ABOUT A WING-AILERON-TAB COMBINATION INCLUDING AERODYNAMIC BALANCE

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SUMMARY

The present paper presents a continuation of the work published in Report No. 496. The results of that paper have been extended to include the effect of aerodynamic balance and the effect of a tab added to the aileron. The aerodynamic coefficients are presented in a form convenient for application to the flutter problem.

INTRODUCTION

It is the object of this paper to present theoretical expressions for the forces and the moments in a uniform horizontal air stream on a plane airfoil performing small sinusoidal motions in several degrees of freedom: vertical motion, torsional movement about an arbitrary spanwise axis, aileron movement about a hinge axis not necessarily located at the leading edge of the aileron, and tab movement similar to the aileron movement. The solution of this problem has direct application to the larger problem of flutter involving these various degrees of freedom and, in particular, to flutter of tails with control surfaces, including servocontrols.

The development of the theory is analogous with that of Theodorsen (reference 1) who treats explicitly the case of three degrees of freedom: vertical motion, torsional movement about an arbitrary spanwise axis, and an aileron movement about a hinge axis located at the leading edge of the aileron.

Since this work was originally begun, there have appeared two German papers, one by Küssner and Schwarz (reference 2) and one by Dietze (reference 3), that bear directly on the problem. A comparison of the results of this paper with the results of Küssner and Schwarz, obtained by a different development, is given in appendix A.

AIR FORCES AND MOMENTS

Figure 1 represents a wing section with two hinges, an aileron (rudder) hinge at $x=e$ and a tab hinge at $x=f$. The leading edge of the wing is at $x=-1$ and the trailing edge at $x=1$. The leading edge of the aileron is at $x=c$ and the distance from the hinge to the aileron leading edge $e-c$ is denoted by l. The leading edge of the tab is at $x=d$ and the distance from the tab hinge to the tab leading edge $f-d$ is denoted by m. The wing is undergoing the following motions with small amplitudes: a displacement h (velocity $\dot{h}$) in a vertical direction downward; a turning about $x=a$, the instantaneous angle of attack being α; a rotation of the aileron about $x=e$, the angle of the aileron or rudder being β measured with respect to the wing; and a rotation of the tab about $x=f$, the angle of the tab being γ measured with respect to the aileron. The

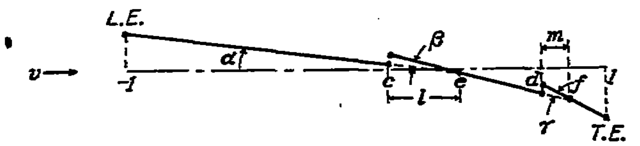


FIGURE 1.—Representation of wing section with aileron and tab showing main parameters.

actual chord is considered to be of length $2b$, so that b is used throughout the analysis as a reference length.

The procedure and method follow those of reference 1. In order to avoid needless repetition of certain expressions contained in this reference, the following notation is frequently used. A symbol or equation followed by (reference 1) denotes the corresponding expression of reference 1. The final results will, however, be explicitly given independently of reference 1.

The air forces and the moments are treated in two groups: the noncirculatory and the circulatory. The expressions for the noncirculatory part consist of apparent-mass terms, which do not depend on the vorticity in the wake. The circulatory part takes account of the vorticity in the wake generated at the trailing edge.

Noncirculatory forces and moments.—The noncirculatory velocity potential at the surface, associated with the various motions of the airfoil, is

$$\phi=\phi_h+\phi_\alpha+\phi_{\dot\alpha}+\phi_\beta+\phi_{\dot\beta}+\phi_\gamma+\phi_{\dot\gamma} \tag{1}$$

where only ϕ_β, $\phi_{\dot\beta}$, ϕ_γ, and $\phi_{\dot\gamma}$ need be given here.[1]

$$\phi_\beta=\phi_\beta\ (\text{reference 1})+v\beta lb\frac{1}{\pi}\log N(x,c) \tag{2}$$

$$\phi_{\dot\beta}=\phi_{\dot\beta}\ (\text{reference 1})$$
$$-\dot\beta lb^2\frac{1}{\pi}\left[\sqrt{1-x^2}\cos^{-1}c-(x-c)\log N(x,c)\right] \tag{3}$$

and

$$\left.\begin{aligned}\phi_\gamma&=\phi_\beta\\ \phi_{\dot\gamma}&=\phi_{\dot\beta}\end{aligned}\right\} \tag{4}$$

with β, c, and l replaced by γ, d, and m, respectively. The two extra terms appearing in ϕ_β and $\phi_{\dot\beta}$, which contain the coefficient l, arise from the aerodynamic-balance effect, that is, the offset of the aileron hinge position from the aileron leading edge. The derivation is as follows:

The motion of the aileron around $x=e$ is considered separated into two parts, a turning around the aileron leading edge plus a vertical displacement of the aileron relative to the wing (fig. 2). The amplitude of the

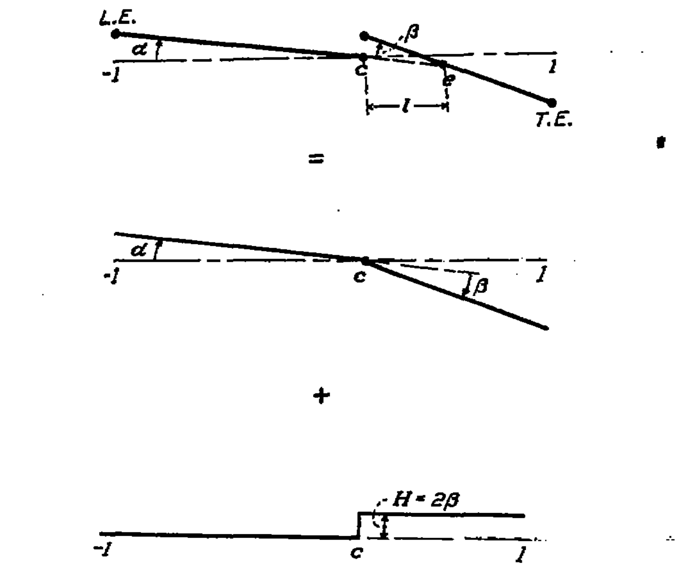


FIGURE 2.—Representation of the motion of the aileron around $x=e$ as separated into a turning around $x=c$ plus a vertical displacement.

first type of motion is β and of the second type of motion is $bH=b\beta(e-c)=b\beta l$. The additional potentials referred to are then due to the effect of the vertical displacement bH and the associated vertical velocity $b\dot H$.

The potential associated with bH is determined from a limiting case (as $c'\rightarrow c$) of a shape (fig. 3) located on the x axis from $x=-1$ to $x=c$ and at ordinate H from $x=c'$ to $x=1$ and joined by a straight-line segment from $x=c$ to $x=c'$. This potential is associated with a vertical-velocity distribution:

$$\begin{aligned}w(x_1)&=0 & -1<x_1<c\\ &=v_H=\frac{vH}{c'-c} & c<x_1<c'\\ &=0 & c'<x_1<1\end{aligned}$$

The surface potential associated with a vertical upward velocity of the air of magnitude w at the element located at $x=x_1$ is

$$\Delta\phi=\frac{bw}{2\pi}\log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}dx_1 \tag{5}$$

where

$$y=\sqrt{1-x^2}\text{ and }y_1=\sqrt{1-x_1^2}$$

Equation (5) is fundamental to the description of the noncirculatory flow pattern since, by integration with respect to x_1, any admissible potential distribution may be obtained. The integrated result desired in this case as $c'\rightarrow c$ is simply

$$\begin{aligned}\phi_H&=\frac{bvH}{2\pi}\log\frac{(x-c)^2+(\sqrt{1-x^2}-\sqrt{1-c^2})^2}{(x-c)^2+(\sqrt{1-x^2}+\sqrt{1-c^2})^2}\\ &=vb\beta l\frac{1}{\pi}\log N(x,c)\end{aligned} \tag{6}$$

where

$$N(x,c)=\left|\frac{1-xc-\sqrt{1-x^2}\sqrt{1-c^2}}{c-x}\right|$$

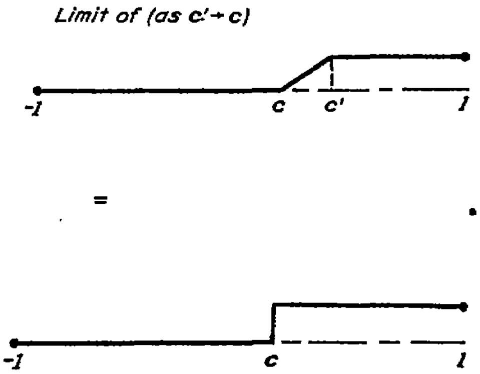


FIGURE 3.—Representation of the sharp vertical displacement as a limit.

The potential associated with $b\dot H$ is due to a vertical velocity distribution

$$\begin{aligned}w&=0 & -1<x_1<c\\ &=\dot\beta lb & c<x_1<1\end{aligned}$$

and is (see ϕ_β, reference 1, p. 5)

$$\begin{aligned}\phi_{\dot H}&=\int_{x_1=-1}^{1}\Delta\phi\\ &=-\dot\beta lb^2\frac{1}{\pi}[\sqrt{1-x^2}\cos^{-1}c-(x-c)\log N(x,c)]\end{aligned} \tag{7}$$

Exactly similar considerations were made for the tab and lead to equations (4).

It may be remarked that the analysis assumes no leak of fluid in the gap between the aileron and the wing; that is, the gap is considered sealed.

The following new sets of integral evaluations will be required. The expressions for the T and Y terms are listed in appendix B. The T terms are functions of c or of d only. When no explicit mention is made, c is

[1] The expressions for the ϕ values are to be understood prefixed by $\pm$ signs: by + for the upper side of the line segment and by − for the lower side.

to be understood. The Y terms are functions of both c and d.

$$\int_c^1 \phi_\beta dx = -\frac{bv\beta}{2\pi}\left(T_5 + 2l\sqrt{1-c^2}\cos^{-1}c\right)$$

$$\int_c^1 \phi_{\dot\beta} dx = -\frac{b^2\dot\beta}{2\pi}\left(T_4 - lT_5\right)$$

$$\int_c^1 \phi_\beta (x-c) dx = -\frac{bv\beta}{2\pi}\left(T_2 - lT_0\right)$$

$$\int_c^1 \phi_{\dot\beta} (x-c) dx = -\frac{b^2\dot\beta}{2\pi}\left(T_3 - lT_4\right)$$

$$\int_1^1 \phi_\beta dx = -\frac{bv\beta}{2}\left(T_4 + 2\sqrt{1-c^2}\right)$$

$$\int_{-1}^1 \phi_{\dot\beta} dx = -\frac{b^2\dot\beta}{2}\left(T_1 - lT_4\right)$$

$$\int_{-1}^1 \phi_\beta (x-c) dx = -\frac{bv\beta}{2}\left(T_8 - lc\sqrt{1-c^2}\right)$$

$$\int_{-1}^1 \phi_{\dot\beta} (x-c) dx = -\frac{b^2\dot\beta}{2}\left(T_7 - lT_8\right)$$

$$\int_c^1 \phi_\gamma dx = -\frac{bv\gamma}{2\pi}\left(Y_1 + mY_2\right)$$

$$\int_c^1 \phi_{\dot\gamma} dx = -\frac{b^2\dot\gamma}{2\pi}\left(Y_3 - mY_1\right)$$

$$\int_c^1 \phi_\gamma (x-c) dx = -\frac{bv\gamma}{2\pi}\left(Y_4 - mY_5\right)$$

$$\int_c^1 \phi_{\dot\gamma} (x-c) dx = -\frac{b^2\dot\gamma}{2\pi}\left(Y_6 - mY_4\right)$$

$$\int_d^1 \phi_\beta dx = -\frac{bv\beta}{2\pi}\left(Y_1 + lY_7\right)$$

$$\int_d^1 \phi_{\dot\beta} dx = -\frac{b^2\dot\beta}{2\pi}\left(Y_4 - lY_1\right)$$

$$\int_d^1 \phi_\beta (x-d) dx = -\frac{bv\beta}{2\pi}\left(Y_3 - lY_8\right)$$

$$\int_d^1 \phi_{\dot\beta} (x-d) dx = -\frac{b^2\dot\beta}{2\pi}\left(Y_6 - lY_3\right)$$

The pressure difference on an element of the airfoil located at x is

$$p = -2\rho\left(\frac{v}{b}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right) \tag{8}$$

and the total force (positive downward) is therefore

$$P = -2\rho b\int_{-1}^1 \dot\phi dx \tag{9}$$

The moment on the airfoil (positive clockwise) about $x=a$ is

$$M_\alpha = b^2\int_{-1}^1 (x-a)p dx$$
$$= -2\rho b\int_{-1}^1 \dot\phi(x-c)dx + 2\rho vb\int_{-1}^1 \phi dx$$
$$-2\rho b^2(c-a)\int_{-1}^1 \dot\phi dx \tag{10}$$

The moment on the aileron (positive clockwise) about the hinge $x=e$ is

$$M_\beta = b^2\int_c^1 (x-e)p dx$$

This moment may be written

$$M_\beta = -2\rho b^2\int_c^1 \dot\phi(x-c)dx + 2\rho vb\int_c^1 \phi dx$$
$$+2\rho b^2 l\int_c^1 \dot\phi dx + 2\rho vbl[\phi]_c^1 \tag{11}$$

Similarly, for the moment on the tab about the hinge $x=f$,

$$M_\gamma = -2\rho b^2\int_d^1 \dot\phi(x-d)dx + 2\rho vb\int_d^1 \phi dx$$
$$+2\rho b^2 m\int_d^1 \dot\phi dx + 2\rho vbm[\phi]_d^1 \tag{12}$$

Circulatory terms.—The potential at the surface of the airfoil associated with an element (counterclockwise) of vorticity in the wake at x_0 of magnitude $\Delta\Gamma = Ubdx_0$ is (reference 1, p. 6)

$$\phi_\Gamma dx_0 = \frac{\Delta\Gamma}{2\pi}\tan^{-1}\frac{\sqrt{1-x^2}\sqrt{x_0^2-1}}{1-xx_0} \tag{13}$$

where $|x|<1$ and $x_0>1$.

The potential for the entire wake is

$$\Phi_\Gamma = \int_1^\infty \phi_\Gamma dx_0 \tag{14}$$

With the assumption that the wake remains where formed, the expression for the pressure at x (equation (8)) becomes

$$p = -2\rho\frac{v}{b}\left(\frac{\partial\Phi_\Gamma}{\partial x} + \frac{\partial\Phi_\Gamma}{\partial x_0}\right)$$
$$= -2\rho v\int_1^\infty \frac{U}{2\pi}\frac{x_0+x}{\sqrt{1-x^2}\sqrt{x_0^2-1}}dx_0 \tag{15}$$

The Kutta condition for smooth flow at the trailing edge, which requires that $\partial/\partial x(\Phi_\Gamma+\phi)$ must remain finite for $x=1$, leads to the result

$$\frac{1}{2\pi}\int_1^\infty U\sqrt{\frac{x_0+1}{x_0-1}}dx_0 = v\alpha + \dot h + b\left(\frac{1}{2}-a\right)\dot\alpha + \frac{1}{\pi}(T_{10}-lT_{21})v\beta$$
$$+\frac{1}{2\pi}(T_{11}-2lT_{10})b\dot\beta + \frac{1}{\pi}(T_{10}(d)-mT_{21}(d))v\gamma$$
$$+\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))b\dot\gamma \tag{16}$$

This quantity will be denoted by Q.

When the various degrees of freedom of the airfoil are undergoing sinusoidal motions of the form $e^{i\omega t}$ for a long period of time, the wake is also sinusoidal and it is convenient to introduce the parameter k defined by the relation

$$\omega t = ks = k\frac{vt}{b}$$

or

$$k = \frac{\omega b}{v} \tag{17}$$

The force (negative lift) on the airfoil due to vorticity is

$$P=b\int_{-1}^{1}p\,dx=-2\pi\rho vbC(k)Q \tag{18}$$

where $C(k)=F(k)+iG(k)$ denotes the fundamental function introduced by Theodorsen (reference 1, p. 8).

The moment about $x=a$ is

$$M_\alpha=b^2\int_{-1}^{1}p(x-a)dx=2\pi\rho vb^2\left[\left(a+\frac{1}{2}\right)C(k)-\frac{1}{2}\right]Q \tag{19}$$

The hinge moment on the aileron about $x=e$ is

$$M_\beta=b^2\int_c^1 p(x-e)dx=b^2\int_c^1 p(x-c)dx+b^2(c-e)\int_c^1 p\,dx$$
$$=-\rho vb^2(T_{12}C(k)-T_4)Q+\rho vb^2l(2T_{20}C(k)+2\sqrt{1-c^2})Q \tag{20}$$

Similarly, the moment on the tab around its hinge, $x=f$, is

$$M_\gamma=-\rho vb^2(T_{12}(d)C(k)-T_4(d))Q+\rho vb^2m(2T_{20}(d)C(k)+2\sqrt{1-d^2})Q \tag{21}$$

Finally, when both the noncirculatory and the circulatory terms are combined, there result the following expressions for the force and moments.

TOTAL FORCE

$$P=-\rho b^2(\pi\ddot{h}+v\pi\dot{\alpha}-\pi ba\ddot{\alpha}-vT_4\dot{\beta}-T_1b\ddot{\beta}-vT_4(d)\dot{\gamma}-T_1(d)b\ddot{\gamma})-\rho b^2l(-2\sqrt{1-c^2}v\dot{\beta}+bT_4\ddot{\beta})$$
$$-\rho b^2m(-2\sqrt{1-d^2}v\dot{\gamma}+bT_4(d)\ddot{\gamma})-2\pi\rho vbCQ \tag{22}$$

where

$$Q=v\alpha+\dot{h}+b\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{1}{\pi}(T_{10}-lT_{21})v\beta+\frac{1}{2\pi}(T_{11}-2lT_{10})b\dot{\beta}+\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))v\gamma+\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))b\dot{\gamma}$$

TOTAL MOMENT ABOUT $x=a$

$$M_\alpha=-\rho b^2\left[-a\pi b\ddot{h}+\pi\left(\frac{1}{2}-a\right)vb\dot{\alpha}+\pi b^2\left(\frac{1}{8}+a^2\right)\ddot{\alpha}+T_{15}v^2\beta+T_{16}vb\dot{\beta}+2T_{13}b^2\ddot{\beta}+T_{15}(d)v^2\gamma+T_{16}(d)vb\dot{\gamma}\right.$$
$$\left.+2T_{13}(d)b^2\ddot{\gamma}\right]-\rho b^2l(T_{22}v^2\beta+T_{23}vb\dot{\beta}+T_{24}b^2\ddot{\beta})-\rho b^2m(T_{22}(d)v^2\gamma+T_{23}(d)vb\dot{\gamma}+T_{24}(d)b^2\ddot{\gamma})+2\pi\rho vb^2\left(a+\frac{1}{2}\right)CQ \tag{23}$$

TOTAL AILERON HINGE MOMENT

$$M_\beta=-\rho b^2\left(-T_1b\ddot{h}+T_{17}vb\dot{\alpha}+2T_{13}b^2\ddot{\alpha}+\frac{1}{\pi}T_{18}v^2\beta+\frac{1}{\pi}T_{19}vb\dot{\beta}-\frac{1}{\pi}T_3b^2\ddot{\beta}+\frac{1}{\pi}Y_9v^2\gamma+\frac{1}{\pi}Y_{10}vb\dot{\gamma}-\frac{1}{\pi}Y_6b^2\ddot{\gamma}\right)$$
$$-\rho b^2l\left(T_4b\ddot{h}+T_{25}vb\dot{\alpha}+T_{24}b^2\ddot{\alpha}+\frac{1}{\pi}T_{26}v^2\beta+\frac{1}{\pi}T_{27}bv\dot{\beta}+\frac{2}{\pi}T_2b^2\ddot{\beta}+\frac{1}{\pi}Y_{11}v^2\gamma+\frac{1}{\pi}Y_{12}bv\dot{\gamma}+\frac{1}{\pi}Y_3b^2\ddot{\gamma}\right)$$
$$-\rho b^2l^2\left(\frac{1}{\pi}T_{28}v^2\beta+\frac{1}{\pi}T_{29}bv\dot{\beta}-\frac{1}{\pi}T_5b^2\ddot{\beta}\right)-\rho b^2m\left(\frac{1}{\pi}Y_{13}v^2\gamma+\frac{1}{\pi}Y_{14}bv\dot{\gamma}+\frac{1}{\pi}Y_4b^2\ddot{\gamma}\right)$$
$$-\rho b^2lm\left(\frac{1}{\pi}Y_{15}v^2\gamma+\frac{1}{\pi}Y_{16}bv\dot{\gamma}-\frac{1}{\pi}Y_1b^2\ddot{\gamma}\right)-\rho vb^2(T_{12}-2lT_{20})CQ \tag{24}$$

TOTAL TAB MOMENT

$$M_\gamma=-\rho b^2\left(-T_1(d)b\ddot{h}+T_{17}(d)vb\dot{\alpha}+2T_{13}(d)b^2\ddot{\alpha}+\frac{1}{\pi}Y_{17}v^2\beta+\frac{1}{\pi}Y_{18}vb\dot{\beta}-\frac{1}{\pi}Y_6b^2\ddot{\beta}+\frac{1}{\pi}T_{18}(d)v^2\gamma+\frac{1}{\pi}T_{19}(d)vb\dot{\gamma}\right.$$
$$\left.-\frac{1}{\pi}T_3(d)b^2\ddot{\gamma}\right)-\rho b^2m\left(T_4(d)b\ddot{h}+T_{25}(d)vb\dot{\alpha}+T_{24}(d)b^2\ddot{\alpha}+\frac{1}{\pi}Y_{19}v^2\beta+\frac{1}{\pi}Y_{20}vb\dot{\beta}+\frac{1}{\pi}Y_4b\ddot{\beta}\right.$$
$$\left.+\frac{1}{\pi}T_{26}(d)v^2\gamma+\frac{1}{\pi}T_{27}(d)bv\dot{\gamma}+\frac{2}{\pi}T_2(d)b^2\ddot{\gamma}\right)-\rho b^2m^2\left(\frac{1}{\pi}T_{28}(d)v^2\gamma+\frac{1}{\pi}T_{29}(d)bv\dot{\gamma}-\frac{1}{\pi}T_5(d)b^2\ddot{\gamma}\right)$$
$$-\rho b^2l\left(\frac{1}{\pi}Y_{21}v^2\beta+\frac{1}{\pi}Y_{22}bv\dot{\beta}+\frac{1}{\pi}Y_3b^2\ddot{\beta}\right)-\rho b^2lm\left(\frac{1}{\pi}Y_{23}v^2\beta+\frac{1}{\pi}Y_{24}bv\dot{\beta}-\frac{1}{\pi}Y_1b^2\ddot{\beta}\right)-\rho vb^2(T_{12}(d)-2mT_{20}(d))CQ \tag{25}$$

Discussion of the term T_{28}.—The concentrated sink-source representing the steep break (fig. 3) properly describes the main flow pattern, but the local flow pattern at the break is incorrect. The underlying theory excludes the possibility of representing the flow at a steep break. The limiting process therefore cannot be used in this simple theory, as far as the local flow is concerned.

There is one term that depends on the local flow condition at the break. This term arises in the evalua-

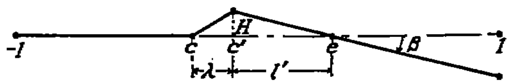


FIGURE 4.—Representation of the effective mean camber line for a displaced aileron with hinge position at $x=e$ as depending upon an additional parameter c'.

tion of $[\phi_\beta]_c^1$ (equation (11)) and is present in the expression for the force on the aileron. It occurs, then, in the expression for hinge moment M_β in the term T_{28} associated with the coefficient $l^2\beta$.

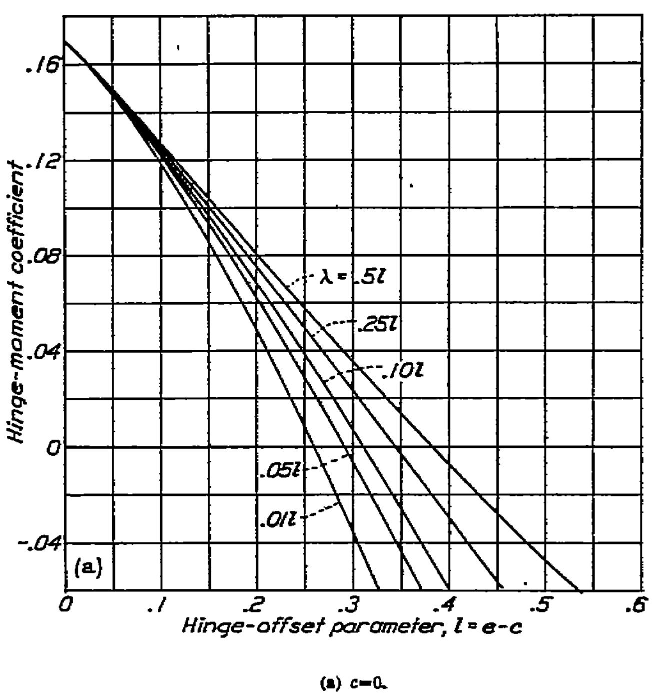
In order to picture the local flow and, at the same time, retain physical reality, it is necessary either to disregard a certain small neighborhood of the break or to spread the concentrated sink-source over a certain finite area. This end may be accomplished by regarding the mean camber line of the displaced aileron with rounded leading edge as depending on an additional parameter c' (fig. 4). Let $e-c'=l'$ and $c'-c=\lambda$. The velocity potential at the surface replacing equation (2) is then

$$\phi_\beta=\phi_{\beta,c'}+\frac{l'}{\lambda}(\phi_{\beta,c'}-\phi_{\beta,c})$$

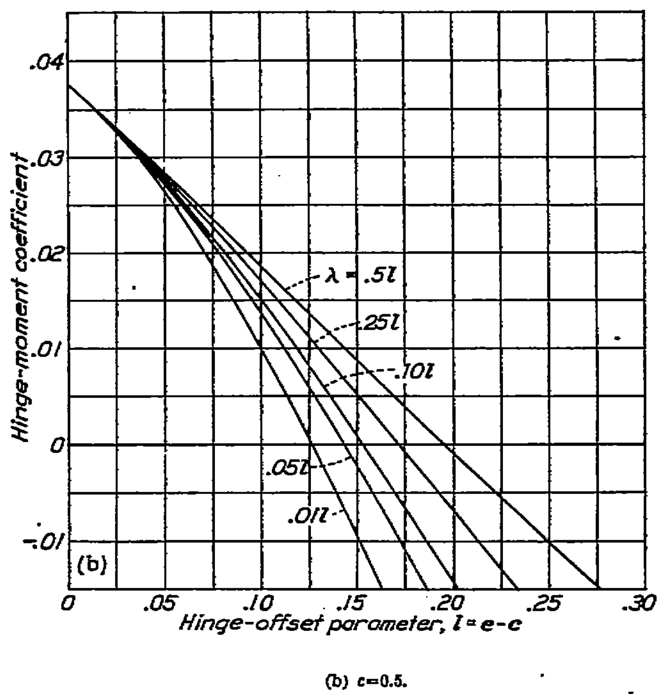
where $\phi_{\beta,c}$ denotes $\frac{v b \beta}{\pi}[\sqrt{1-x^2}\cos^{-1}c-(x-c)\log N(x,c)]$

The analysis can be performed by use of this equation and a similar one for $\phi_{\dot\beta}$ instead of equations (2) and (3). The result will, however, differ essentially from that already presented in two respects: (1) The average value of l will be slightly less than $e-c$ and will be nearly $e-\frac{c+c'}{2}$. (2) The term $\log N(c,c')$ will occur in T_{28} to replace the infinite term $\log N(c,c)$.

An effective value of c' may be estimated in any given case or may be determined by experiment. The essential point is that the difference $c'-c$ cannot become zero. It appears probable that $c'-c$ is greater than, say, 0.05 $(e-c)$ and less than 0.40 $(e-c)$. As an average value 0.25 $(e-c)$ appears good. Within these limits the value of the term $\log N(c,c')$ is fairly independent of the selection of c'. The effect of the choice of c' in the steady case on the hinge moment is illustrated in figure 5. Conversely, the experimental hinge-moment values may be used to estimate c'.



(a) $c=0$.



(b) $c=0.5$.

FIGURE 5.—Hinge-moment coefficient against hinge-offset parameter $l=e-c$ for various values of the mean-camber-line parameter $\lambda=c'-c$. (Steady case, $\frac{1}{k}=0$.) The ordinate is

$$-M_\beta/\pi\rho b^2v^2\beta=\frac{1}{\pi^2}(T_{13}+T_{16}T_{12})+l\frac{1}{\pi^2}(T_{26}-2T_{24}T_{26}-T_{12}T_{21})+l^2\frac{1}{\pi^2}(T_{28}+2T_{24}T_{11})$$

Equations (22) to (25) may be conveniently expressed in a coefficient form:

$$P=-\pi\rho\omega^2b^3\left(\frac{h}{b}A_{ch}+\alpha A_{c\alpha}+\beta A_{c\beta}+\gamma A_{c\gamma}\right) \quad (22')$$

$$M_\alpha=-\pi\rho\omega^2b^4\left(\frac{h}{b}A_{ah}+\alpha A_{a\alpha}+\beta A_{a\beta}+\gamma A_{a\gamma}\right) \quad (23')$$

$$M_\beta=-\pi\rho\omega^2b^4\left(\frac{h}{b}A_{bh}+\alpha A_{b\alpha}+\beta A_{b\beta}+\gamma A_{b\gamma}\right) \quad (24')$$

$$M_\gamma=-\pi\rho\omega^2b^4\left(\frac{h}{b}A_{dh}+\alpha A_{d\alpha}+\beta A_{d\beta}+\gamma A_{d\gamma}\right) \quad (25')$$

where

$$\omega=kv/b, \quad A_{ch}=R_{ch}+iI_{ch}, \; A_{c\alpha}=R_{c\alpha}+iI_{c\alpha}, \text{ etc.}$$

and where

$$R_{a\alpha}=-A_{\alpha1}+\left(\frac{1}{4}-a^2\right)\frac{2G}{k}-\left(\frac{1}{2}+a\right)\frac{2F}{k^2}$$

$$R_{a\beta}=-A_{\beta1}+\frac{1}{k^2}A_{\beta3}+\left(\frac{1}{2}+a\right)\left[\left(\frac{T_{11}-2lT_{10}}{2\pi}\right)\frac{2G}{k}-\left(\frac{T_{10}-lT_{21}}{\pi}\right)\frac{2F}{k^2}\right]$$

$$R_{ah}=-A_{h1}+\left(\frac{1}{2}+a\right)\frac{2G}{k}$$

$$R_{a\gamma}=-A_{\gamma1}+\frac{1}{k^2}A_{\gamma3}+\left(\frac{1}{2}+a\right)\left[\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))\frac{2G}{k}-\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2F}{k^2}\right]$$

$$R_{b\alpha}=-B_{\alpha1}-\frac{1}{2\pi}(T_{12}-2lT_{20})\left[\left(\frac{1}{2}-a\right)\frac{2G}{k}-\frac{2F}{k^2}\right]$$

$$R_{b\beta}=-B_{\beta1}+\frac{1}{k^2}B_{\beta3}-\frac{1}{2\pi}(T_{12}-2lT_{20})\left[\frac{1}{2\pi}(T_{11}-2lT_{10})\frac{2G}{k}-\frac{1}{\pi}(T_{10}-lT_{21})\frac{2F}{k^2}\right]$$

$$R_{bh}=-B_{h1}-\frac{1}{2\pi}(T_{12}-2lT_{20})\frac{2G}{k}$$

$$R_{b\gamma}=-B_{\gamma1}+\frac{1}{k^2}B_{\gamma3}-\frac{1}{2\pi}(T_{12}-2lT_{20})\left[\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))\frac{2G}{k}-\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2F}{k^2}\right]$$

$$R_{c\alpha}=-C_{\alpha1}-\left(\frac{1}{2}-a\right)\frac{2G}{k}+\frac{2F}{k^2}$$

$$R_{c\beta}=-C_{\beta1}-\frac{1}{2\pi}(T_{11}-2lT_{10})\frac{2G}{k}+\frac{1}{\pi}(T_{10}-lT_{21})\frac{2F}{k^2}$$

$$R_{ch}=-C_{h1}-\frac{2G}{k}$$

$$R_{c\gamma}=-C_{\gamma1}-\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))\frac{2G}{k}+\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2F}{k^2}$$

$$R_{d\alpha}=-D_{\alpha1}-\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\left[\left(\frac{1}{2}-a\right)\frac{2G}{k}-\frac{2F}{k^2}\right]$$

$$R_{d\beta}=-D_{\beta1}+\frac{1}{k^2}D_{\beta3}-\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\left[\frac{1}{2\pi}(T_{11}-2lT_{10})\frac{2G}{k}-\frac{1}{\pi}(T_{10}-lT_{21})\frac{2F}{k^2}\right]$$

$$R_{dh}=-D_{h1}-\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\frac{2G}{k}$$

$$R_{d\gamma}=-D_{\gamma1}+\frac{1}{k^2}D_{\gamma3}-\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\left[\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))\frac{2G}{k}-\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2F}{k^2}\right]$$

$$I_{a\alpha}=\frac{1}{k}\left[A_{\alpha2}-\left(\frac{1}{2}+a\right)\frac{2G}{k}-\left(\frac{1}{4}-a^2\right)2F\right]$$

$$I_{a\beta}=\frac{1}{k}\left\{A_{\beta2}-\left(\frac{1}{2}+a\right)\left[\frac{1}{\pi}(T_{10}-lT_{21})\frac{2G}{k}+\frac{1}{2\pi}(T_{11}-2lT_{10})2F\right]\right\}$$

$$I_{ah}=\frac{1}{k}\left[-\left(\frac{1}{2}+a\right)2F\right]$$

$$I_{a\gamma}=\frac{1}{k}\left\{A_{\gamma2}-\left(\frac{1}{2}+a\right)\left[\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2G}{k}+\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))2F\right]\right\}$$

$$I_{b\alpha}=\frac{1}{k}\left\{B_{\alpha 2}+\frac{1}{2\pi}(T_{12}-2lT_{20})\left[\frac{2G}{k}+\left(\frac{1}{2}-a\right)2F\right]\right\}$$

$$I_{b\beta}=\frac{1}{k}\left\{B_{\beta 2}+\frac{1}{2\pi}(T_{12}-2lT_{20})\left[\frac{1}{\pi}(T_{10}-lT_{21})\frac{2G}{k}+\frac{1}{2\pi}(T_{11}-2lT_{10})2F\right]\right\}$$

$$I_{bh}=\frac{1}{k}\frac{1}{2\pi}(T_{12}-2lT_{20})2F$$

$$I_{b\gamma}=\frac{1}{k}\left\{B_{\gamma 2}+\frac{1}{2\pi}(T_{12}-2lT_{20})\left[\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2G}{k}+\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))2F\right]\right\}$$

$$I_{c\alpha}=\frac{1}{k}\left[C_{\alpha 2}+\frac{2G}{k}+\left(\frac{1}{2}-a\right)2F\right]$$

$$I_{c\beta}=\frac{1}{k}\left[C_{\beta 2}+\frac{1}{\pi}(T_{10}-lT_{21})\frac{2G}{k}+\frac{1}{2\pi}(T_{11}-2lT_{10})2F\right]$$

$$I_{ch}=\frac{1}{k}2F$$

$$I_{c\gamma}=\frac{1}{k}\left[C_{\gamma 2}+\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2G}{k}+\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))2F\right]$$

$$I_{d\alpha}=\frac{1}{k}\left\{D_{\alpha 2}+\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\left[\frac{2G}{k}+\left(\frac{1}{2}-a\right)2F\right]\right\}$$

$$I_{d\beta}=\frac{1}{k}\left\{D_{\beta 2}+\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\left[\frac{1}{\pi}(T_{10}-lT_{21})\frac{2G}{k}+\frac{1}{2\pi}(T_{11}-2lT_{10})2F\right]\right\}$$

$$I_{dh}=\frac{1}{k}\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))2F$$

$$I_{d\gamma}=\frac{1}{k}\left\{D_{\gamma 2}+\frac{1}{2\pi}(T_{12}(d)-2mT_{20}(d))\left[\frac{1}{\pi}(T_{10}(d)-mT_{21}(d))\frac{2G}{k}+\frac{1}{2\pi}(T_{11}(d)-2mT_{10}(d))2F\right]\right\}$$

$$A_{\alpha 1}=\frac{1}{8}+a^2$$

$$A_{\alpha 2}=\frac{1}{2}-a$$

$$A_{\beta 1}=\frac{2T_{13}}{\pi}+l\frac{T_{24}}{\pi}$$

$$A_{\beta 2}=\frac{1}{\pi}(T_{16}+lT_{23})$$

$$A_{\beta 3}=\frac{1}{\pi}(T_{15}+lT_{22})$$

$$A_{h1}=-a$$

$$A_{\gamma 1}=\frac{1}{\pi}(2T_{13}(d)+mT_{24}(d))$$

$$A_{\gamma 2}=\frac{1}{\pi}(T_{16}(d)+mT_{23}(d))$$

$$A_{\gamma 3}=\frac{1}{\pi}(T_{15}(d)+mT_{22}(d))$$

$$B_{\alpha 1}=A_{\beta 1}$$

$$B_{\alpha 2}=\frac{1}{\pi}(T_{17}+lT_{25})$$

$$B_{\beta 1}=\frac{1}{\pi^2}(-T_3+2lT_2-l^2T_5)$$

$$B_{\beta 2}=\frac{1}{\pi^2}(T_{19}+lT_{27}+l^2T_{29})$$

$$B_{\beta 3}=\frac{1}{\pi^2}(T_{18}+lT_{26}+l^2T_{28})$$

$$B_{h1}=\frac{1}{\pi}(-T_1+lT_4)$$

$$B_{\gamma 1}=\frac{1}{\pi^2}(-Y_6+lY_3+mY_4-lmY_1)$$

$$B_{\gamma 2}=\frac{1}{\pi^2}(Y_{10}+lY_{12}+mY_{14}+lmY_{16})$$

$$B_{\gamma 3}=\frac{1}{\pi^2}(Y_9+lY_{11}+mY_{13}+lmY_{15})$$

$$C_{\alpha 1}=A_{h1}$$
$$C_{\alpha 2}=1$$
$$C_{\beta 1}=B_{h1}$$
$$C_{\beta 2}=\frac{1}{\pi}(-T_4-2l\sqrt{1-c^2})$$
$$C_{h1}=1$$
$$C_{\gamma 1}=D_{h1}$$
$$C_{\gamma 2}=\frac{1}{\pi}(-T_4(d)-2m\sqrt{1-d^2})$$

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$$D_{\alpha 1}=A_{\gamma 1}$$
$$D_{\alpha 2}=\frac{1}{\pi}(T_{17}(d)+mT_{25}(d))$$
$$D_{\beta 1}=B_{\gamma 1}$$
$$D_{\beta 2}=\frac{1}{\pi^2}(Y_{18}+mY_{20}+lY_{22}+lmY_{24})$$
$$D_{\beta 3}=\frac{1}{\pi^2}(Y_{17}+mY_{19}+lY_{21}+lmY_{23})$$
$$D_{h1}=\frac{1}{\pi}(-T_1(d)+mT_4(d))$$
$$D_{\gamma 1}=\frac{1}{\pi^2}(-T_3(d)+2mT_2(d)-m^2T_5(d))$$
$$D_{\gamma 2}=\frac{1}{\pi^2}(T_{19}(d)+mT_{27}(d)+m^2T_{29}(d))$$
$$D_{\gamma 3}=\frac{1}{\pi^2}(T_{18}(d)+mT_{26}(d)+m^2T_{28}(d))$$

CONCLUDING REMARKS

The material presented in the preceding pages represents an extension of the work published in reference 1, which has been expanded to include the tab functions and the effect of aerodynamic balance. Inasmuch as this addition fits in with the general arrangement of the earlier report, reference should be made to that report, and also to reference 4, for the application to the flutter problem.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTE FOR AERONAUTICS,
LANGLEY FIELD, VA., *December 16, 1941.*

APPENDIX A

COMPARISON WITH REFERENCE 2

In order to compare the results of this report with those of Küssner and Schwarz (reference 2), the following relationships are noted:

$$c=-\cos\varphi$$
$$d=-\cos\psi$$
$$k=-i\omega$$
$$C(k)=1+2T(-i\omega)$$
$$a=-\frac{1}{2}$$
$$T_1=-\frac{1}{2}\Phi_4$$
$$T_2=-\frac{1}{2}\Phi_{37}$$
$$T_3=-\frac{1}{4}\Phi_{12}$$
$$T_4=-\Phi_3$$
$$T_5=-\Phi_{17}$$
$$T_9=\frac{1}{8}\Phi_6$$
$$T_{10}=\Phi_1$$
$$T_{11}=\Phi_2$$
$$T_{12}=\Phi_8$$
$$T_{13}=\frac{1}{8}\Phi_7$$
$$T_{15}=\Phi_5$$
$$T_{16}=\frac{1}{2}\Phi_6$$
$$T_{17}=\frac{1}{2}\Phi_9$$
$$T_{18}=\Phi_{10}$$
$$T_{19}=\frac{1}{2}\Phi_{11}$$
$$T_{20}=\Phi_{31}$$
$$T_{21}=\Phi_{13}$$
$$T_{22}=-\Phi_{15}$$
$$T_{23}=-2\Phi_5$$
$$T_{24}=-\frac{1}{4}\Phi_6$$
$$T_{25}=-\Phi_{32}$$
$$T_{26}=-(\Phi_{18}+\Phi_{35})^1$$
$$T_{27}=-(\Phi_{19}+\Phi_{36})$$
$$T_{28}=\pi^2 r_d^*$$
$$T_{29}=\Phi_{16}$$

$$\log N(c,d)=-L(\varphi,\psi)$$
$$Y_1(c,d)=-X_{14}(\varphi,\psi)$$
$$Y_2(c,d)=Y_{13}(c,d)-T_4(c)T_{21}(d)$$
$$Y_3(c,d)=-X_{18}(\varphi,\psi)$$
$$Y_4(c,d)=-X_5(\varphi,\psi)$$
$$Y_5(c,d)=Y_6(d,c)=X_2(\varphi,\psi)-X_{17}(\varphi,\psi)$$
$$Y_6(c,d)=-X_{10}(\varphi,\psi)$$
$$Y_7(c,d)=X_3(\varphi,\psi)+\Phi_{14}(\varphi)\Phi_{31}(\psi)$$
$$Y_{17}(c,d)=Y_9(d,c)=X_8(\varphi,\psi)$$
$$Y_{18}(c,d)=Y_{10}(d,c)=X_9(\varphi,\psi)$$
$$Y_{19}(c,d)=Y_{11}(d,c)=-X_3(\varphi,\psi)$$
$$Y_{20}(c,d)=Y_{12}(d,c)=-X_4(\varphi,\psi)$$
$$Y_{21}(c,d)=Y_{13}(d,c)=-X_{16}(\varphi,\psi)$$
$$Y_{22}(c,d)=Y_{14}(d,c)=-X_{17}(\varphi,\psi)$$
$$Y_{23}(c,d)=Y_{15}(d,c)=X_{12}(\varphi,\psi)$$
$$Y_{24}(c,d)=Y_{16}(d,c)=X_{13}(\varphi,\psi)$$

[1] There appears to be an error in the sign of the numerical values for Φ_{18} in table 3 of reference 2.

APPENDIX B

EXPRESSION FOR THE T AND Y FUNCTIONS

T FUNCTIONS

$$T_0=c\sqrt{1-c^2}\cos^{-1}c-(1-c^2)$$
$$=\sqrt{1-c^2}\,(c\cos^{-1}c-\sqrt{1-c^2})$$

$$T_1=-\frac{1}{3}(2+c^2)\sqrt{1-c^2}+c\cos^{-1}c$$

$$T_2=c(1-c^2)-(1+c^2)\sqrt{1-c^2}\cos^{-1}c+c(\cos^{-1}c)^2$$

$$T_3=-\frac{1}{8}(1-c^2)(5c^2+4)+\frac{1}{4}c(7+2c^2)\sqrt{1-c^2}\cos^{-1}c$$
$$-\left(\frac{1}{8}+c^2\right)(\cos^{-1}c)^2$$

$$T_4=c\sqrt{1-c^2}-\cos^{-1}c$$

$$T_5=-(1-c^2)+2c\sqrt{1-c^2}\cos^{-1}c-(\cos^{-1}c)^2$$

$$T_6=T_2$$

$$T_7=-\frac{1}{8}c(7+2c^2)\sqrt{1-c^2}-\left(\frac{1}{8}+c^2\right)\cos^{-1}c$$

$$T_8=-\frac{1}{3}(1+2c^2)\sqrt{1-c^2}+c\cos^{-1}c=-\frac{1}{3}(1-c^2)^{3/2}-cT_4$$

$$T_9=\frac{1}{2}\left[\frac{1}{3}(1-c^2)^{3/2}+aT_4\right]$$

$$T_{10}=\sqrt{1-c^2}+\cos^{-1}c$$

$$T_{11}=(2-c)\sqrt{1-c^2}+(1-2c)\cos^{-1}c$$

$$T_{12}=(2+c)\sqrt{1-c^2}-(1+2c)\cos^{-1}c$$

$$T_{13}=-\frac{1}{2}(T_7+(c-a)T_1)$$

$$T_{14}=\frac{1}{16}+\frac{1}{2}ac$$

$$T_{15}=T_4+T_{10}=(1+c)\sqrt{1-c^2}$$

$$T_{16}=T_1-T_8-(c-a)T_4+\frac{1}{2}T_{11}=\frac{2}{3}(1-c^2)^{3/2}-\left(\frac{1}{2}-a\right)T_4$$

$$T_{17}=-2T_9-T_1+\left(a-\frac{1}{2}\right)T_4=-\frac{1}{3}(1-c^2)^{3/2}-T_1-\frac{1}{2}T_4$$

$$T_{18}=T_5-T_4T_{10}$$

$$T_{19}=-\frac{1}{2}T_4T_{11}$$

$$T_{20}=-\sqrt{1-c^2}+\cos^{-1}c$$

$$T_{21}=\sqrt{\frac{1+c}{1-c}}$$

$$T_{22}=2\sqrt{1-c^2}-\sqrt{\frac{1+c}{1-c}}$$

$$T_{23}=(-1-2c+2a)\sqrt{1-c^2}$$

$$T_{24}=T_8+(c-a)T_4=-2T_9$$

$$T_{25}=T_4-(1-c)\sqrt{1-c^2}$$

$$T_{26}=2\sqrt{1-c^2}T_{20}+T_4\sqrt{\frac{1+c}{1-c}}$$

$$T_{27}=T_4T_{10}-\sqrt{1-c^2}T_{11}$$

$$T_{28}=2(1+c+\log N(c,c'))$$

$$T_{29}=2\sqrt{1-c^2}T_{10}$$

The term T_{28} is discussed separately in the paper.. The variable c is to be understood when no explicit variable is indicated for the T terms.

Y FUNCTIONS

$$Y_1(c,d)=-\sqrt{1-c^2}\sqrt{1-d^2}-\cos^{-1}c\cos^{-1}d$$
$$+d\sqrt{1-d^2}\cos^{-1}c+c\sqrt{1-c^2}\cos^{-1}d-(d-c)^2\log N(c,d)$$

where

$$N(c,d)=\left|\frac{1-cd-\sqrt{1-c^2}\sqrt{1-d^2}}{d-c}\right|$$

$$Y_2(c,d)=2\sqrt{1-d^2}\cos^{-1}c-2(d-c)\log N(c,d)$$

$$Y_3(c,d)=\frac{1}{3}(c+2d)\sqrt{1-c^2}\sqrt{1-d^2}+d\cos^{-1}c\cos^{-1}d$$
$$-\frac{1}{3}(2+d^2)\sqrt{1-d^2}\cos^{-1}c$$
$$-\frac{1}{3}(1+3cd-c^2)\sqrt{1-c^2}\cos^{-1}d$$
$$+\frac{1}{3}(d-c)^3\log N(c,d)$$

$$Y_4(c,d)=Y_3(d,c)$$
$$=\frac{1}{3}(d+2c)\sqrt{1-c^2}\sqrt{1-d^2}+c\cos^{-1}c\cos^{-1}d$$
$$-\frac{1}{3}(2+c^2)\sqrt{1-c^2}\cos^{-1}d$$
$$-\frac{1}{3}(1+3cd-d^2)\sqrt{1-d^2}\cos^{-1}c$$
$$-\frac{1}{3}(d-c)^3\log N(c,d)$$

$$Y_5(c,d)=-\sqrt{1-c^2}\sqrt{1-d^2}+(2c-d)\sqrt{1-d^2}\cos^{-1}c$$
$$+(d-c)^2\log N(c,d)$$

$$Y_6(c,d)=-\frac{1}{2}\sqrt{1-c^2}\sqrt{1-d^2}\left(1+\frac{1}{6}c^2+\frac{1}{6}d^2+\frac{11}{12}cd\right)$$
$$-\left(\frac{1}{8}+cd\right)\cos^{-1}c\cos^{-1}d$$
$$+\frac{1}{3}\left[\frac{d}{4}\left(\frac{5}{2}-d^2\right)+c(2+d^2)\right]\sqrt{1-d^2}\cos^{-1}c$$
$$+\frac{1}{3}\left[\frac{c}{4}\left(\frac{5}{2}-c^2\right)+d(2+c^2)\right]\sqrt{1-c^2}\cos^{-1}d$$
$$+\frac{(d-c)^4}{12}\log N(c,d)$$

$$Y_7(c,d)=2\sqrt{1-c^2}\cos^{-1} d+2(d-c)\log N(c,d)$$
$$=Y_2(d,c)$$
$$Y_8(c,d)=-\sqrt{1-c^2}\sqrt{1-d^2}+(2d-c)\sqrt{1-c^2}\cos^{-1} d$$
$$+(d-c)^2\log N(c,d)=Y_8(d,c)$$
$$Y_9(c,d)=Y_1-T_4(c)T_{10}(d)$$
$$Y_{10}(c,d)=Y_3-Y_4-\frac{1}{2}T_4(c)T_{11}(d)$$
$$Y_{11}(c,d)=Y_7-2\sqrt{1-c^2}T_{10}(d)$$
$$Y_{12}(c,d)=Y_1-Y_8-\sqrt{1-c^2}T_{11}(d)$$
$$Y_{13}(c,d)=Y_2+T_4(c)T_{21}(d)$$
$$Y_{14}(c,d)=Y_8-Y_9$$
$$Y_{15}(c,d)=2\sqrt{1-c^2}T_{21}(d)+2\log N(c,d)$$
$$Y_{16}(c,d)=Y_2-Y_7+2\sqrt{1-c^2}T_{10}(d)$$
$$Y_{17}(c,d)=Y_1-T_4(d)T_{10}(c)=Y_9(d,c)$$
$$Y_{18}(c,d)=Y_4-Y_3-\frac{1}{2}T_4(d)T_{11}(c)=Y_{10}(d,c)$$
$$Y_{19}(c,d)=Y_2-2\sqrt{1-d^2}T_{10}=Y_{11}(d,c)$$
$$Y_{20}(c,d)=Y_1-Y_8-\sqrt{1-d^2}T_{11}=Y_{12}(d,c)$$
$$Y_{21}(c,d)=Y_7+T_4(d)T_{21}(c)=Y_{13}(d,c)$$
$$Y_{22}(c,d)=-Y_1+Y_8+T_{10}(c)T_4(d)=Y_{14}(d,c)$$
$$Y_{23}(c,d)=2\sqrt{1-d^2}T_{21}(c)+2\log N(c,d)=Y_{15}(d,c)$$
$$Y_{24}(c,d)=Y_7-Y_2+2\sqrt{1-d^2}T_{10}=Y_{16}(d,c)$$

In the evaluation of the Y terms the following pertinent integrals occur:

$$\int (x-c)^n \log N\,dx=\frac{(x-c)^{n+1}}{n+1}\log N-\frac{\sqrt{1-c^2}}{n+1}\int\frac{(x-c)^n}{\sqrt{1-x^2}}dx$$

where

$$N=\left|\frac{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}{x-c}\right|$$

In order to evaluate the last integral put $x=\cos\theta$; for example, consider $n=1$:

$$\int_d^1 (x-c)\log N\,dx=-\frac{(d-c)^2}{2}\log N(c,d)-\frac{\sqrt{1-c^2}}{2}\left(\sqrt{1-d^2}-c\cos^{-1} d\right)$$

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